CS 1190

Fundamentals of Quantum Information Science

Lecture 4

Superposition and Entanglement

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Contents

1 What Are We Learning Today?

In our earlier lectures, we introduced the idea of a **qubit**. Today, we will go one level deeper. Our main question is:

What information is stored in a qubit, and what changes when two qubits become entangled?

We will build the ideas in this order:

$$\boxed{\text{Qubit} \to \text{Superposition} \to \text{Amplitude} \to \text{Probability} \to \text{Phase} \to \text{Interference} \to \text{Two Qubits} \to \text{Entanglement}}$$

We will repeatedly ask three questions:

1. **What does this state mean?**
2. **What happens if we measure it?**
3. **What happens after the next quantum gate?**

1.1 Learning Objectives

By the end of today's class, you should be able to:

- explain what a qubit represents;
- read a simple quantum state written using bra-ket notation;
- explain superposition using both words and equations;
- distinguish probability amplitude from probability;
- explain why the number $1/\sqrt{2}$ appears so often;
- calculate simple measurement probabilities;
- explain why phase matters;
- predict the action of a Hadamard gate;
- identify the four computational basis states of two qubits;
- explain the basic action of a CNOT gate;
- distinguish superposition from entanglement; and
- follow the creation of a Bell pair step by step.

2 Start With a Classical Bit

A classical computer stores information using **bits**. A bit can have one of two values: 0 or 1. At the time we read the bit, the result is one of these two values.

Think Before Reading Further

Suppose you are given one classical bit. What can you read from it? $\boxed{0 \text{ or } 1}$
Now suppose you are given two classical bits. What combinations are possible? $00, \quad 01, \quad 10, \quad 11.$
Keep these four combinations in mind. We will return to them later.

3 The Quantum Bit

The basic unit of quantum information is the **quantum bit**, or **qubit**. We write the two computational basis states as $|0\rangle$ and $|1\rangle$.

The notation $|\ \rangle$ is called **ket notation**. Thus, $|0\rangle$ is pronounced "ket zero," and $|1\rangle$ is pronounced "ket one."

Bra-ket notation is associated with the mathematical language developed by Paul Dirac for quantum mechanics [1].

3.1 What Can Physically Represent a Qubit?

A qubit is quantum information. Different physical systems can store that information. Examples include:

- photon polarization;
- electron spin;
- superconducting circuits;
- trapped ions; and
- neutral atoms.

In this course, we will often use **photons** as our intuitive example. For example, we may define $|0\rangle \longleftrightarrow$ horizontal polarization and $|1\rangle \longleftrightarrow$ vertical polarization.
Equivalently, $|0\rangle \equiv |H\rangle$, $|1\rangle \equiv |V\rangle$.

Key Idea

- A qubit is not a particular particle.
- A qubit is a unit of quantum information.
- Different physical quantum systems can be used to represent that information.

4 A Qubit Can Be More Than $|0\rangle$ or $|1\rangle$

A qubit does not have to be only $|0\rangle$ or only $|1\rangle$. A general pure qubit state can be written as

$$\boxed{|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle\,.}$$

The symbol

$$|\psi\rangle$$

simply means

the quantum state we are describing.

The quantities α and β are called **probability amplitudes** [3].

5 Amplitude Is Not Probability

Suppose

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle\,.$$

The amplitudes are α and β. The probabilities are obtained from the squared magnitudes:

$$\boxed{P(0) = |\alpha|^2} \qquad \boxed{P(1) = |\beta|^2.}$$

This relationship is part of the standard measurement rule of quantum mechanics, associated historically with Max Born [2].

Because a computational-basis measurement must return either 0 or 1, $P(0) + P(1) = 1.$
Therefore,

$$\boxed{|\alpha|^2 + |\beta|^2 = 1.}$$

This is called the **normalization condition**.

Worked Example: Finding Measurement Probabilities

Consider

$$|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle.$$

The amplitude of $|0\rangle$ is

$$\frac{\sqrt{3}}{2}.$$

Therefore,

$$P(0) = \left|\frac{\sqrt{3}}{2}\right|^2 = \frac{3}{4}.$$

Thus,

$$\boxed{P(0) = 75\%.}$$

For $|1\rangle$,

$$P(1) = \left|\frac{1}{2}\right|^2 = \frac{1}{4}.$$

Thus,

$$\boxed{P(1) = 25\%.}$$

Check:

$$\frac{3}{4} + \frac{1}{4} = 1.$$

Your Turn: Calculate the Probabilities

Consider

$$|\psi\rangle = \frac{3}{5}|0\rangle + \frac{4}{5}|1\rangle.$$

Calculate

$$P(0)$$

and

$$P(1).$$

Then check that

$$P(0) + P(1) = 1.$$

Do the calculation before reading further.

We obtain

$$P(0) = \left(\frac{3}{5}\right)^2 = \frac{9}{25},$$

and

$$P(1) = \left(\frac{4}{5}\right)^2 = \frac{16}{25}.$$

Therefore,

$$\frac{9}{25} + \frac{16}{25} = \frac{25}{25} = 1.$$

6 Why Do We Keep Seeing $1/\sqrt{2}$?

One expression appears constantly in quantum computing:

$$\frac{1}{\sqrt{2}}.$$

Why?

Let us derive it instead of memorizing it. Suppose we want a qubit whose two basis states have equal amplitudes:

$$|\psi\rangle = a\,|0\rangle + a\,|1\rangle\,.$$

Normalization requires

$$|a|^2 + |a|^2 = 1.$$

Therefore,

$$2|a|^2 = 1.$$

So,

$$|a|^2 = \frac{1}{2}.$$

Taking the positive square root,

$$a = \frac{1}{\sqrt{2}}.$$

Therefore,

$$\boxed{|\psi\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}.}$$

Now check the probabilities:

$$P(0) = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2},$$

and

$$P(1) = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}.$$

Thus,

$$\boxed{P(0) = P(1) = 50\%.}$$

Key Idea

The factor

$$\frac{1}{\sqrt{2}}$$

is not an arbitrary quantum rule. It appears because the total probability must equal 1.

Your Turn: Is This a Valid Qubit State?

Consider

$$|\psi\rangle = \frac{1}{2}|0\rangle + \frac{1}{2}|1\rangle.$$

Is this state normalized?
Calculate

$$\left|\frac{1}{2}\right|^2 + \left|\frac{1}{2}\right|^2.$$

We obtain

$$\frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

Therefore,

$$\boxed{\frac{1}{2}|0\rangle + \frac{1}{2}|1\rangle}$$

is **not** normalized.
For equal amplitudes, we need

$$\boxed{\frac{1}{\sqrt{2}}}$$

rather than

$$\frac{1}{2}.$$

7 What Does Measurement Mean?

Suppose

$$|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle.$$

We found $P(0) = 75\%$ and $P(1) = 25\%$.
Does one measurement give us "75% of 0"? No.

A single computational-basis measurement gives one classical result: 0 or 1. The probabilities become visible when the same state is prepared and measured repeatedly.

For example, after approximately 1000 repetitions, we might observe roughly 750 zeros and 250 ones. The exact numbers need not be exactly 750 and 250.

Key Idea

Quantum probabilities describe the statistics of repeated experiments.
A single measurement still gives one definite classical outcome.

8 Superposition

Now we can define one of the central ideas of quantum information. Suppose

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle\,.$$

If both amplitudes are nonzero, then, with respect to the computational basis, the state is a **superposition** of $|0\rangle$ and $|1\rangle$.

A particularly important example is

$$\boxed{|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}.}$$

8.1 What Does Superposition Mean?

You may sometimes hear:

"A qubit is both 0 and 1 at the same time."

This phrase may help build an initial intuition, but it can also create confusion.

A better description is:

Key Idea

A qubit in superposition has probability amplitudes associated with multiple basis states. When we measure the qubit, one outcome is observed according to probabilities determined by those amplitudes.

9 Classical Uncertainty Is Not the Same as Superposition

Imagine that a coin is hidden under a cup. The coin is either H or T. You may not know which one.

But your lack of knowledge does not mean the coin is physically in a quantum superposition. It already has one classical value.

Now compare that with

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}.$$

The important difference is that quantum amplitudes can later **interfere**. That interference can change future measurement probabilities.

Classical ignorance about a hidden coin does not behave this way.

10 The Hadamard Gate

One of the most important gates in quantum computing is the **Hadamard gate**, written as H.

Two important superposition states are

$$\boxed{|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}} \qquad \text{and} \qquad \boxed{|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}.}$$

The Hadamard gate transforms the computational basis states into these superposition states:

$$\boxed{H\,|0\rangle = |+\rangle} \qquad \text{and} \qquad \boxed{H\,|1\rangle = |-\rangle\,.}$$

Worked Example: From Certainty to Superposition

Start with

$$|0\rangle .$$

If we measure immediately,

$$P(0) = 100\%.$$

Now apply H:

$$|0\rangle \xrightarrow{H} |+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}.$$

If we measure now,

$$P(0) = 50\% \qquad \text{and} \qquad P(1) = 50\%.$$

The Hadamard gate has changed a definite computational-basis state into an equal superposition.

Your Turn: Predict Before Reading

A qubit starts in

$$|1\rangle .$$

Apply a Hadamard gate.
What state do you obtain?

Using

$$H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}},$$

we obtain

$$\boxed{|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}.}$$

If we measure immediately in the computational basis,

$$P(0) = 50\%, \qquad P(1) = 50\%.$$

11 Photon Example of Superposition

Recall our photon encoding:

$$|0\rangle = |H\rangle, \qquad |1\rangle = |V\rangle .$$

Here, $|H\rangle$ denotes horizontal polarization and $|V\rangle$ denotes vertical polarization.
Then

$$|+\rangle = \frac{|H\rangle + |V\rangle}{\sqrt{2}}$$

corresponds to a diagonal polarization state.
Similarly,

$$|-\rangle = \frac{|H\rangle - |V\rangle}{\sqrt{2}}$$

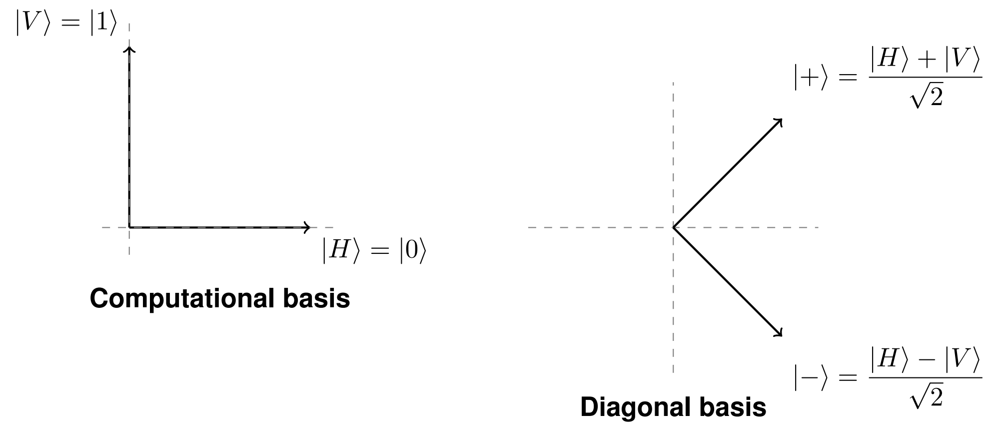

Figure 1: Photon polarization picture of superposition. Horizontal and vertical polarization form one basis, while diagonal polarizations form another basis.

corresponds to the orthogonal diagonal polarization state.

The photon example gives us a physical picture of superposition. Horizontal and vertical polarization form one basis, while diagonal polarizations form another basis, as shown in Figure 1.

12 Phase: Why the Plus and Minus Signs Matter

Now compare

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \qquad \text{and} \qquad |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}.$$

If we immediately measure either state in the computational basis,

$$P(0) = 50\% \qquad \text{and} \qquad P(1) = 50\%.$$

So we should ask:

If their probabilities are identical, why should we care about the minus sign?

The answer is **phase**.

The plus and minus signs represent different relative phases between the amplitudes.

Key Idea

Two quantum states can give the same immediate measurement probabilities but behave differently in later quantum operations. The reason is that amplitudes contain phase information.

13 Interference

To understand phase, think about waves. Two waves can reinforce each other. This is called

$$\boxed{\text{constructive interference.}}$$

Two waves can also cancel each other. This is called

$$\boxed{\text{destructive interference.}}$$

Quantum amplitudes can also interfere.

This interference is one of the central mechanisms used by quantum algorithms [3].

Worked Example: The Difference Between $|+\rangle$ and $|-\rangle$

Recall that

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

and

$$|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}.$$

We also know that the Hadamard gate acts on the computational basis states as

$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

and

$$H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}.$$

Now let us see what happens when we apply a Hadamard to $|+\rangle$ and $|-\rangle$.
Applying H to $|+\rangle$, Start with

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}.$$

Apply H:

$$H|+\rangle = H\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right).$$

Because a quantum gate acts on each part of a superposition,

$$H|+\rangle = \frac{1}{\sqrt{2}}\left(H|0\rangle + H|1\rangle\right).$$

Now substitute the action of the Hadamard gate:

$$= \frac{1}{\sqrt{2}}\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} + \frac{|0\rangle - |1\rangle}{\sqrt{2}}\right).$$

Combine the factors of $1/\sqrt{2}$:

$$= \frac{1}{2}\left(|0\rangle + |1\rangle + |0\rangle - |1\rangle\right).$$

Now look carefully at the two basis-state amplitudes.
For $|1\rangle$, $|1\rangle - |1\rangle = 0$.. They cancel.
For $|0\rangle$, $|0\rangle + |0\rangle = 2|0\rangle$. They reinforce.
Therefore,

$$H|+\rangle = \frac{1}{2}\left(2|0\rangle\right) = \boxed{|0\rangle}.$$

So,

$$\boxed{H|+\rangle = |0\rangle.}$$

Worked Example: Continue

Applying H to $|-\rangle$. Now start with

$$|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}.$$

Apply H:

$$H\,|-\rangle = H\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right).$$

Therefore,

$$H\,|-\rangle = \frac{1}{\sqrt{2}}\left(H\,|0\rangle - H\,|1\rangle\right).$$

Substitute again:

$$= \frac{1}{\sqrt{2}}\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} - \frac{|0\rangle - |1\rangle}{\sqrt{2}}\right).$$

Simplifying,

$$= \frac{1}{2}\left(|0\rangle + |1\rangle - |0\rangle + |1\rangle\right).$$

Now look at the amplitudes.
For $|0\rangle$, $|0\rangle - |0\rangle = 0$. They cancel.
For $|1\rangle$, $|1\rangle + |1\rangle = 2\,|1\rangle$. They reinforce.
Therefore,

$$H\,|-\rangle = \frac{1}{2}\left(2\,|1\rangle\right) = \boxed{|1\rangle}.$$

So,

$$\boxed{H\,|-\rangle = |1\rangle\,.}$$

What Did We Learn?
Before applying the Hadamard,

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \qquad \text{and} \qquad |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

both give

$$P(0) = 50\% \qquad \text{and} \qquad P(1) = 50\%$$

if measured directly in the computational basis.
However, the two states have different relative phases.
After applying H,

$$\boxed{|+\rangle \xrightarrow{H} |0\rangle} \qquad \text{while} \qquad \boxed{|-\rangle \xrightarrow{H} |1\rangle\,.}$$

The different signs determine which amplitudes reinforce and which cancel.
This is an example of **quantum interference**.

14 A Surprising Two-Hadamard Experiment

Start with $|0\rangle$. Apply one Hadamard:

$$|0\rangle \xrightarrow{H} |+\rangle \, .$$

If we measured now, we would obtain 0 or 1 with equal probability.

But suppose we do **not** measure. Instead, apply a second Hadamard: $|+\rangle \xrightarrow{H} |0\rangle$.
Therefore,

$$\boxed{H^2 |0\rangle = |0\rangle \, .}$$

This is an important lesson.

The intermediate state had two nonzero amplitudes, but those amplitudes could interfere during the second Hadamard operation.

Think Before Reading Further

If the state after the first Hadamard were merely a classical $50/50$ lottery between 0 and 1, would a second Hadamard always return us to $|0\rangle$?
This is one reason superposition should not be understood as ordinary classical uncertainty.

In-Class Quiz: Superposition and Interference

A qubit begins in $|0\rangle$.
We apply $H \to H \to$ Measure.
What result do we obtain ideally?

A. 0 with probability 50%

B. 1 with probability 50%

C. 0 with probability 100%

D. 0 and 1 with equal probability

15 Pause and Review

At this point, we have learned four important ideas.

1. A qubit can be written as $\alpha |0\rangle + \beta |1\rangle$.
2. Measurement probabilities are

$$|\alpha|^2 \qquad \text{and} \qquad |\beta|^2.$$

3. A qubit can be in a superposition.
4. Phase allows amplitudes to interfere.

Now we are ready to move from

$$\boxed{\text{one qubit}} \quad \rightarrow \quad \boxed{\text{two qubits}}.$$

16 From One Qubit to Two Qubits

One classical bit has two possible values: $0, \quad 1.$

Two classical bits have four possible combinations: $00, \quad 01, \quad 10, \quad 11.$

Two qubits also have four computational basis states:

$$\boxed{|00\rangle\,, \quad |01\rangle\,, \quad |10\rangle\,, \quad |11\rangle\,.}$$

For example, $|01\rangle$ means: first qubit is 0; second qubit is 1.

Your Turn: Read the Two-Qubit State

What does $|10\rangle$ mean?

- What is the first qubit?
- What is the second qubit?

The answer is: $\boxed{\text{first qubit} = 1}$ and $\boxed{\text{second qubit} = 0}$.

17 Combining Two Qubits

When we combine quantum systems, we use the symbol $\otimes$. It is called the **tensor product**.

For today's lecture, we can read the symbol simply as:

combine these two quantum systems.

For example, $|0\rangle \otimes |0\rangle = |00\rangle\,.$

Similarly,

$$|0\rangle \otimes |1\rangle = |01\rangle\,, \qquad |1\rangle \otimes |0\rangle = |10\rangle\,, \qquad \text{and} \qquad |1\rangle \otimes |1\rangle = |11\rangle\,.$$

We do not need the full matrix calculation of tensor products today.

18 A General Two-Qubit State

A general pure two-qubit state can be written as $|\psi\rangle = \alpha\,|00\rangle + \beta\,|01\rangle + \gamma\,|10\rangle + \delta\,|11\rangle\,.$

The amplitudes must satisfy

$$|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1.$$

The corresponding measurement probabilities are

$$P(00) = |\alpha|^2, \quad P(01) = |\beta|^2, \quad P(10) = |\gamma|^2, \text{and} \quad P(11) = |\delta|^2.$$

Worked Example: A Two-Qubit Measurement

Consider

$$|\psi\rangle = \frac{|00\rangle + |01\rangle}{\sqrt{2}}.$$

The two possible measurement results are 00 and 01. Each has probability $\frac{1}{2}$. Therefore, $P(00) = 50\%$, and $P(01) = 50\%$.

Notice something interesting: the first qubit is always 0. Only the second qubit changes.

19 Superposition Does Not Automatically Mean Entanglement

Consider

$$\frac{|00\rangle + |10\rangle}{\sqrt{2}}.$$

This is clearly a superposition of two computational basis states.
But watch what happens when we rewrite it:

$$\frac{|00\rangle + |10\rangle}{\sqrt{2}} = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes |0\rangle .$$

Using

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}},$$

we obtain

$$\boxed{|+\rangle \otimes |0\rangle .}$$

This tells us exactly what each qubit is doing.
The first qubit is

$$|+\rangle .$$

The second qubit is

$$|0\rangle .$$

We can describe them separately.
Therefore, the state is **not entangled**.

Key Idea

A two-qubit state can contain superposition without being entangled.
The key question is:

Can we describe each qubit with its own independent state?

If yes, the pure state is separable.
If no, the pure state is entangled.

20 Now Look at an Entangled State

Consider

$$\boxed{\frac{|00\rangle + |11\rangle}{\sqrt{2}}.}$$

Can we write this as

$$|\text{state of qubit 1}\rangle \otimes |\text{state of qubit 2}\rangle ?$$

For this state, we cannot.
That is what makes it entangled.

20.1 A Useful Comparison

You might wonder:

Could both qubits simply be in the state $|+\rangle$?

Let us test that idea.
If both qubits were independently in $|+\rangle$, we would have $|+\rangle \otimes |+\rangle$.
Since

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}},$$

we obtain

$$|+\rangle \otimes |+\rangle = \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) \otimes \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right).$$

Expanding gives

$$\boxed{\frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right).}$$

This is **not**

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}.$$

The terms $|01\rangle$ and $|10\rangle$ are missing from the Bell state. Therefore, the Bell state cannot simply be understood as two independent $|+\rangle$ qubits.

Key Idea

In an entangled pure state, the information belongs to the **joint state of the pair**.
We cannot fully describe the state by assigning one independent pure state to qubit 1 and another independent pure state to qubit 2.

21 The Bell State

The state

$$\boxed{|\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}}$$

is one of four important two-qubit states called **Bell states** [3].
If we measure both qubits in the computational basis,

$$P(00) = \frac{1}{2},$$

$$P(11) = \frac{1}{2},$$

and

$$P(01) = 0,$$

$$P(10) = 0.$$

Therefore, ideally, we see only

$$00$$

or

$$11.$$

22 Alice and Bob

Suppose Alice receives the first qubit.
Bob receives the second.
Together, the pair is in the state

$$|\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}.$$

Alice's individual result is random.
She measures

$$0$$

with probability 50%, or

$$1$$

with probability 50%.
Bob also sees a $50/50$ distribution locally.
However, when they later compare their computational-basis measurement results, they find matching outcomes:

$$00$$

or

$$11.$$

Common Misconception

Alice cannot choose whether her measurement result will be 0 or 1.
Therefore, entanglement does not allow Alice to choose a bit and instantly send that bit to Bob.
Entanglement by itself does not enable faster-than-light communication.

23 Correlation Is Not Automatically Entanglement

Suppose someone prepares two classical bits as

$$00$$

half the time and

$$11$$

half the time.
Alice receives one bit and Bob receives the other.
Their results will also match.
Therefore,

Key Idea

Seeing only 00 and 11 in one measurement basis is not enough, by itself, to establish the uniquely quantum nature of entanglement.

The deeper distinction appears when measurements are made in different bases.

John Bell showed that quantum mechanics predicts correlations that can violate constraints obeyed by local hidden-variable theories [6].

We will not derive Bell's inequality today.

For now, remember:

$$\boxed{\text{entanglement is deeper than “the two answers match.”}}$$

24 How Do We Create Entanglement?

We will now create

$$\left|\Phi^+\right\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

from

$$|00\rangle .$$

We need only two gates:

1. Hadamard H;
2. controlled-NOT, or CNOT.

25 Step 1: Create Superposition

Start with

$$|00\rangle .$$

Apply a Hadamard to the first qubit:

$$|00\rangle \longrightarrow \frac{|00\rangle + |10\rangle}{\sqrt{2}}.$$

Why?

Because the first qubit changes as

$$|0\rangle \xrightarrow{H} \frac{|0\rangle + |1\rangle}{\sqrt{2}},$$

while the second qubit remains

$$|0\rangle .$$

Therefore,

$$\frac{|00\rangle + |10\rangle}{\sqrt{2}} = |+\rangle \otimes |0\rangle .$$

At this moment,

$$\boxed{\text{superposition: yes}}$$

but

$$\boxed{\text{entanglement: no.}}$$

Think Before Reading Further

This step is important.
The Hadamard created superposition, but the two qubits are still separable.
We can still say:

$$\text{qubit 1} = |+\rangle\,,$$

$$\text{qubit 2} = |0\rangle\,.$$

26 Step 2: Apply CNOT

CNOT means

controlled-NOT.

It acts on two qubits.
One qubit is the

$$\boxed{\text{control}}$$

and the other is the

$$\boxed{\text{target}}.$$

For today's circuit, the first qubit is the control.
The rule is:

- if the control is 0, leave the target unchanged;
- if the control is 1, flip the target.

26.1 CNOT Examples

If the input is

$$|00\rangle\,,$$

the control is 0, so nothing changes:

$$\boxed{|00\rangle \to |00\rangle\,.}$$

If the input is

$$|01\rangle\,,$$

the control is still 0:

$$\boxed{|01\rangle \to |01\rangle\,.}$$

If the input is

$$|10\rangle\,,$$

the control is 1, so the target flips:

$$\boxed{|10\rangle \to |11\rangle\,.}$$

If the input is

$$|11\rangle\,,$$

the control is 1, so the target flips:

$$\boxed{|11\rangle \to |10\rangle\,.}$$

26.2 CNOT Table

Input	Control	Output
$\|00\rangle$	0	$\|00\rangle$
$\|01\rangle$	0	$\|01\rangle$
$\|10\rangle$	1	$\|11\rangle$
$\|11\rangle$	1	$\|10\rangle$

Your Turn: Predict CNOT

Predict the output before reading the answer.

$$|11\rangle \xrightarrow{\text{CNOT}} ?$$

Remember:

control $= 1$ means flip the target.

Therefore,

$$\boxed{|11\rangle \to |10\rangle\,.}$$

27 Step 3: CNOT Acts on the Superposition

Before CNOT, we have

$$\frac{|00\rangle + |10\rangle}{\sqrt{2}}.$$

Apply CNOT to each part.
First,

$$|00\rangle \to |00\rangle\,.$$

Second,

$$|10\rangle \to |11\rangle\,.$$

Therefore,

$$\boxed{\frac{|00\rangle + |10\rangle}{\sqrt{2}} \longrightarrow \frac{|00\rangle + |11\rangle}{\sqrt{2}}.}$$

We have created a Bell state.

28 Follow the Whole Circuit

Start with

$$\boxed{|00\rangle}.$$

Apply H to qubit 1:

$$\Downarrow$$

$$\boxed{\frac{|00\rangle + |10\rangle}{\sqrt{2}}}$$

which is a superposition but not entangled.
Then apply CNOT:

$$\Downarrow$$

$$\boxed{\frac{|00\rangle + |11\rangle}{\sqrt{2}}}$$

which is entangled.
Conceptually,

$$\boxed{\text{definite state} \to \text{superposition} \to \text{entanglement.}}$$

29 The Most Important Comparison Today

State	Superposition?	Entangled?
$\lvert 00\rangle$	No	No
$\frac{\lvert 00\rangle + \lvert 10\rangle}{\sqrt{2}}$	Yes	No
$\frac{\lvert 00\rangle + \lvert 11\rangle}{\sqrt{2}}$	Yes	Yes

Key Idea

Superposition and entanglement are related, but they are not the same.
A single qubit can be in superposition.
Entanglement describes a joint quantum state of multiple systems that cannot be separated into independent subsystem states.

30 Your Turn: Follow the Circuit Yourself

Your Turn: Bell-Pair Construction

Start with

$$|00\rangle .$$

Step 1
Apply H to the first qubit.
Write the resulting state.

Step 2
Apply CNOT with qubit 1 as control and qubit 2 as target.
Write the final state.

Step 3
List the possible computational-basis measurement outcomes and their probabilities.

31 What Happens If We Repeat the Measurement?

Suppose we prepare

$$|\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

and measure 1000 times.
Ideally, we expect approximately

$$500$$

measurements of

$$00$$

and approximately

$$500$$

measurements of

$$11.$$

We ideally expect no

$$01$$

or

$$10.$$

On real hardware, noise may introduce additional outcomes.

In-Class Quiz: Predict the Histogram

Which ideal histogram best matches

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}?$$

A

$$25\%\ 00, \quad 25\%\ 01, \quad 25\%\ 10, \quad 25\%\ 11$$

B

$$50\%\ 00, \quad 50\%\ 11$$

C

$$100\%\ 00$$

D

$$50\%\ 01, \quad 50\%\ 10$$

32 In-Class Quiz

In-Class Quiz: Question 1: Probability

Consider

$$|\psi\rangle = \frac{3}{5}|0\rangle + \frac{4}{5}|1\rangle.$$

What is

$$P(1)?$$

A. $\frac{4}{5}$

B. $\frac{16}{25}$

C. $\frac{3}{5}$

D. $\frac{9}{25}$

In-Class Quiz: Question 2: Normalization

Which state gives equal $50/50$ probabilities?

A.

$$\frac{1}{2}|0\rangle + \frac{1}{2}|1\rangle$$

B.

$$\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

C.

$$|0\rangle + |1\rangle$$

In-Class Quiz: Question 3: Hadamard

A qubit starts in

$$|0\rangle .$$

After one Hadamard, it becomes

A. $|1\rangle$

B.
$$\frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

C. $|00\rangle$

D.
$$\frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

In-Class Quiz: Question 4: CNOT

What is

$$|10\rangle \xrightarrow{\text{CNOT}} ?$$

A. $|00\rangle$

B. $|01\rangle$

C. $|10\rangle$

D. $|11\rangle$

In-Class Quiz: Question 5: Superposition or Entanglement?

Which state is entangled?

A.
$$|00\rangle$$

B.
$$\frac{|00\rangle + |10\rangle}{\sqrt{2}}$$

C.
$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

33 Common Misconceptions

33.1 Misconception 1

Common Misconception

Incorrect: A qubit in superposition is literally half 0 and half 1.

Better:
The state contains probability amplitudes associated with the basis states.
A measurement produces one classical outcome.

33.2 Misconception 2

Common Misconception

Incorrect: Superposition is just classical uncertainty.

Better:
Quantum amplitudes contain phase and can interfere.
A hidden classical value does not behave the same way.

33.3 Misconception 3

Common Misconception

Incorrect: Any two-qubit superposition is entangled.

Better:
For example,

$$\frac{|00\rangle + |10\rangle}{\sqrt{2}} = |+\rangle \otimes |0\rangle$$

is a superposition but is not entangled.

33.4 Misconception 4

Common Misconception

Incorrect: If two measurement results always match, the systems must be entangled.

Better:
Classical systems can also be correlated.
Bell-type experiments use multiple measurement settings to distinguish quantum correlations from local classical explanations.

33.5 Misconception 5

Common Misconception

Incorrect: Entanglement allows faster-than-light communication.

Better:
Individual local measurement results are not controllable.
Entanglement therefore cannot by itself transmit a chosen classical message faster than light.

34 What Should You Remember?

If you remember only five things from today's lecture, remember these.

1. A Qubit Can Be Written as

$$\boxed{|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle\,.}$$

2. Probability Comes From the Amplitude

$$\boxed{P(0) = |\alpha|^2, \qquad P(1) = |\beta|^2.}$$

3. Equal Superposition Uses

$$\boxed{\frac{1}{\sqrt{2}}.}$$

4. Superposition Is Not Entanglement

$$\boxed{\frac{|00\rangle + |10\rangle}{\sqrt{2}}}$$

   is a superposition but is not entangled.

5. A Bell State Is Entangled

$$\boxed{|\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}.}$$

35 Concept Map

Start with

$$|0\rangle\,.$$

Apply a Hadamard:

$$\Downarrow$$

$$\frac{|0\rangle + |1\rangle}{\sqrt{2}}.$$

This introduces

$$\boxed{\text{superposition.}}$$

The coefficients are

$$\boxed{\text{probability amplitudes.}}$$

Their squared magnitudes give

$$\boxed{\text{measurement probabilities.}}$$

Their relative phases influence

$$\boxed{\text{interference.}}$$

Now use two qubits:

$$|00\rangle .$$

Apply H to qubit 1:

$$\Downarrow$$

$$\frac{|00\rangle + |10\rangle}{\sqrt{2}}.$$

Apply CNOT:

$$\Downarrow$$

$$\boxed{\frac{|00\rangle + |11\rangle}{\sqrt{2}}}$$

which is an

$$\boxed{\text{entangled Bell state.}}$$

36 Exit Questions

Before leaving class, make sure you can answer these questions without looking back at the notes.

1. What is the difference between an amplitude and a probability?
2. Why does $1/\sqrt{2}$ appear in an equal superposition?
3. Why is

$$\frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

   different from a classical $50/50$ coin flip?
4. What does a Hadamard gate do to $|0\rangle$?
5. What does CNOT do when its control qubit is 1?
6. Why is

$$\frac{|00\rangle + |10\rangle}{\sqrt{2}}$$

   not entangled?

7. Why is

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

different?

37 Answer Key

Probability Practice

For

$$|\psi\rangle = \frac{3}{5}|0\rangle + \frac{4}{5}|1\rangle,$$

$$P(0) = \frac{9}{25},$$

and

$$P(1) = \frac{16}{25}.$$

Normalization Practice

$$\frac{1}{2}|0\rangle + \frac{1}{2}|1\rangle$$

is not normalized because

$$\frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

For equal probabilities, use

$$\frac{1}{\sqrt{2}}.$$

Hadamard Quiz

The correct answer is

$$\boxed{\text{C: } 0 \text{ with probability } 100\%}.$$

because

$$H^2|0\rangle = |0\rangle.$$

CNOT Practice

$$|00\rangle \to |00\rangle,$$

$$|01\rangle \to |01\rangle,$$

$$|10\rangle \to |11\rangle,$$

$$|11\rangle \to |10\rangle.$$

Bell-Pair Construction

Start with

$$|00\rangle .$$

After Hadamard on qubit 1:

$$\frac{|00\rangle + |10\rangle}{\sqrt{2}}.$$

After CNOT:

$$\boxed{\frac{|00\rangle + |11\rangle}{\sqrt{2}}.}$$

Measurement gives

$$00$$

with probability 50%, and

$$11$$

with probability 50%.

In-Class Quiz Answers

1. Probability question:

   $$\boxed{\text{B}}$$

   because

   $$\left|\frac{4}{5}\right|^2 = \frac{16}{25}.$$

2. Equal superposition:

   $$\boxed{\text{B}}$$

   because

   $$\left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}.$$

3. Hadamard:

   $$\boxed{\text{B}}$$

4. CNOT:

   $$\boxed{\text{D}}$$

5. Entangled state:

   $$\boxed{\text{C}}$$

38 Historical Context

The idea of entanglement developed through several important milestones.

1. **1935: Einstein, Podolsky, and Rosen**

   Albert Einstein, Boris Podolsky, and Nathan Rosen published a famous paper questioning whether quantum mechanics provided a complete description of physical reality [4].

   Their argument focused attention on the behavior of separated quantum systems.

2. **1935: Schrödinger**

   Erwin Schrödinger analyzed the unusual relationship between separated quantum systems and emphasized the importance of what we now call entanglement [5].

3. **1964: Bell**

   John Bell showed that local hidden-variable theories must satisfy constraints that quantum mechanics can violate [6]. This result became known as **Bell's theorem**.

4. **1969: CHSH** Clauser, Horne, Shimony, and Holt developed an experimentally useful Bell-type inequality [7].

5. **1982: Aspect and Colleagues** Alain Aspect, Jean Dalibard, and Gérard Roger reported an influential experimental Bell test with photons [8].

6. **2022 Nobel Prize**

   The 2022 Nobel Prize in Physics recognized Alain Aspect, John F. Clauser, and Anton Zeilinger for experiments with entangled photons, tests of Bell inequalities, and contributions to quantum information science [9].

39 Why Entanglement Matters

Entanglement is used in many areas of quantum information science. Examples include:

- quantum teleportation;
- superdense coding;
- quantum communication;
- quantum networking;
- quantum cryptography; and
- multi-qubit quantum algorithms.

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