CS 1190 – Fundamentals of Quantum Information Science

Quantum Teleportation and the Quantum Internet

Why quantum networks need teleportation, and how the protocol works

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The question for today

Imagine two quantum computers in different laboratories. One laboratory has a qubit that the other must use to continue a computation. Sending ordinary 0s and 1s is not enough to describe a single unknown quantum state, and a traveling quantum carrier can be lost. **How can we move the state without measuring it to learn its amplitudes or making a forbidden copy?**

The word *teleportation* can suggest that an object disappears and reappears somewhere else. Here, the quantity transferred is a **quantum state**. Bob's physical carrier already exists. Alice does not keep an additional copy of the arbitrary input state, and the protocol does not provide faster-than-light communication [1].

Our running example is photon polarization. We use $|0\rangle$ for horizontal polarization and $|1\rangle$ for vertical polarization. The same qubit mathematics also applies when the physical carriers are other two-state quantum systems. The circuit below is an ideal logical description; building its operations with photons requires an appropriate optical implementation.

Learning goals

By the end of this lesson, you should be able to explain **why** a quantum internet is useful, distinguish **entanglement swapping** from **state teleportation**, sketch Alice's and Bob's actions, calculate the four correction branches, and describe why teleportation does not copy a qubit or permit faster-than-light messaging.

How to use these notes. Sections in Part I explain the motivation without difficult mathematics. Part II develops the complete circuit and worked calculations. Part III returns to real applications. Short checkpoint quizzes follow the major ideas; a comprehensive quiz and its worked answer key appear at the end.

Part I. Why teleportation is needed

Start with the real communication problem, the quantum internet, and the physical resource requirements.

1 The Quantum Internet: the bigger picture

Think about the internet you already use. Your laptop sends classical packets across fiber links and routers. A router may read a bit, store it, copy it, and send it onward. That works because the information in a classical bit can be copied.

Now consider a network of quantum processors, sensors, and quantum memories. These systems would sometimes need to *share entanglement* or transfer quantum states, rather than merely exchange ordinary numbers. A **quantum internet** is the vision of connecting such devices with quantum communication capabilities, while ordinary classical connections continue to provide control and coordination [7, 6].

Question	Classical internet	Quantum internet
What is exchanged?	Classical bit strings and packets	Entangled resources and quantum states, alongside classical messages
What can intermediate nodes do?	Copy, amplify, and regenerate bits	Cannot clone arbitrary unknown states; need quantum-aware operations
Typical equipment	Hosts, switches, routers	Quantum nodes, photonic links, quantum memories, repeaters (in advanced networks)
What might it enable?	Email, video, distributed classical services	Networked quantum computation, certain secure communication tasks, distributed sensing

Where teleportation fits. An advanced quantum network needs ways to establish entanglement across nodes. Once two endpoints share entanglement, the standard teleportation protocol can transfer an input qubit's state from one endpoint to another using local operations and two classical bits. *Teleportation is an enabling primitive, not the entire quantum internet.* A quantum internet also requires physical links, error management, routing or scheduling, and often quantum memories. Figure 1 gives a high-level architectural view that we will refer back to later. Its cryptographic applications require the security assumptions and verification procedures of the particular protocol, such as QKD; entanglement alone is not an automatic guarantee of security.

A distinction worth remembering

Quantum key distribution (QKD) is one family of quantum communication applications. It can be implemented using prepare-and-measure methods without teleportation. Teleportation instead transfers an arbitrary quantum *state*. A future quantum internet may support both, but they are not the same protocol [6].

Our roadmap. First, understand what goes wrong when we try to copy, measure, or send an unknown qubit. Next, learn how a shared Bell pair and two ordinary bits solve the ideal state-transfer task. Finally, return to the repeater and computing applications with the mathematics in hand.

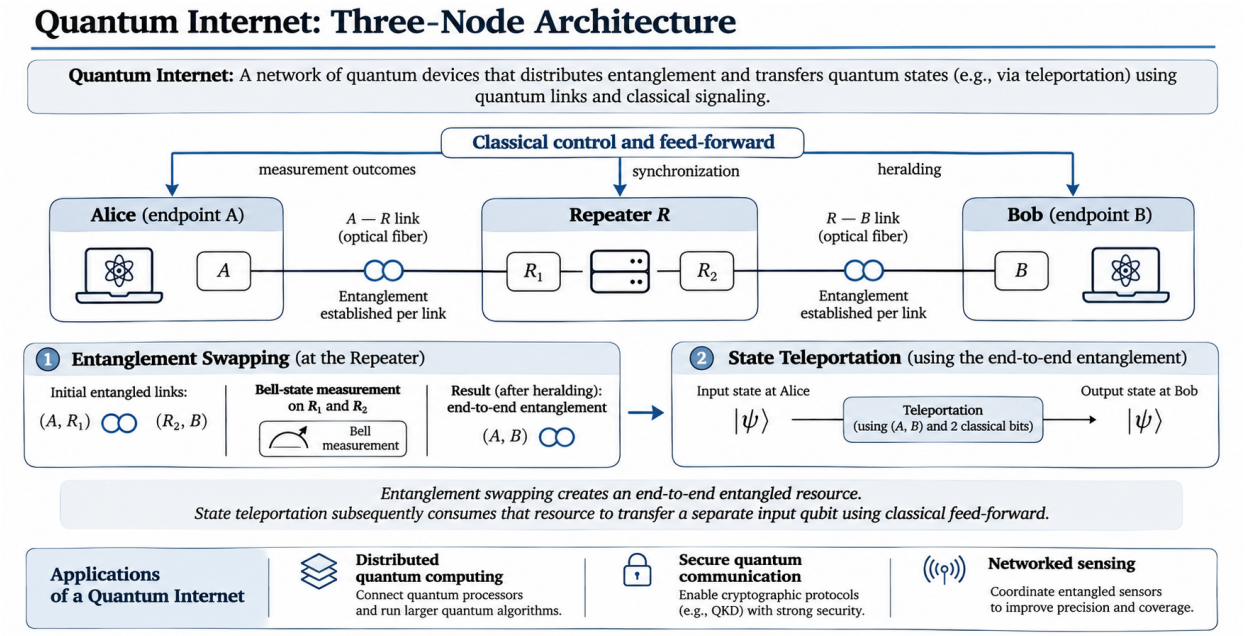


Figure 1: Three-node quantum-internet architecture. The two adjacent links establish (A, R_1) and (R_2, B) entanglement. A Bell-state measurement on R_1 and R_2, together with classical outcome tracking or corrections, can prepare an entangled pair across Alice and Bob. That pair can then be consumed to teleport a separate input state. The diagram depicts a possible idealized architecture, not an already deployed universal quantum internet.

2 Why ordinary messages are not enough

Imagine an ordinary computer sends a photograph to another computer. It converts the photograph into a sequence of 0s and 1s. The receiver reads those bits and reconstructs an image. The sender may keep a copy, and the message can be copied while it travels. This is how the familiar internet works.
Now imagine two *quantum processors*. Alice's processor has a qubit that is partway through a quantum computation. Bob's processor must continue the computation using that same quantum state. The input was produced by quantum operations, so Alice may not know its amplitudes or relative phase. If she measures it to obtain an ordinary 0 or 1, the measurement generally changes what she wanted to send.

The problem teleportation solves

Goal. Transfer an arbitrary quantum state from a qubit at Alice's location to a different qubit at Bob's location, even when neither person knows what that state is.
Challenge. Alice cannot read out one unknown qubit as a complete classical description and cannot make an independent perfect copy of it. A traveling physical qubit may also be lost or disturbed along a connection.
Tool. A previously shared entangled pair lets Alice perform a local measurement and communicate a short correction instruction. After Bob acts on that instruction, his qubit carries the input state [1, 5].

2.1 A situation that ordinary bits cannot handle

Suppose Alice holds

$$|\psi\rangle_S = \alpha|0\rangle + \beta|1\rangle, \qquad |\alpha|^2 + |\beta|^2 = 1. \tag{1}$$

Alice does not know α or β. If she measures in the 0/1 basis, she gets one ordinary bit, but that bit does not provide the full preparation instructions. For example, $|+\rangle$ and $|-\rangle$ both give 0 or 1 with 50/50 probabilities, yet they are different states. We will show the calculation later.
If Alice already knows the preparation recipe, she can describe it to Bob, who can prepare the same *known* state. This is not the difficult case. Teleportation is important when the state is genuinely unknown, perhaps because it is the result of a prior quantum computation or is entangled with something else.

Application scenario A. Two computers collaborating

Alice computes on her own qubits and creates a state she cannot simply inspect without disturbing. Bob has a second quantum device that can perform the next step. The computers need to pass the *quantum state*, not a screenshot of a measurement result. Teleportation is one way of connecting their computations using a shared Bell pair and classical communication. The original information is no longer available as an extra copy at Alice after the operation.

Think and check. If Alice can send millions of ordinary bits to Bob, can she always communicate one *single unknown* qubit by first measuring it?
Explanation. No. Sending many bits is not the obstacle. The measurement does not reveal the entire unknown state from one available copy; it generally changes the state.

3 Why direct quantum transmission is difficult

One obvious approach is to place Alice's photon on an optical fiber. If it arrives with its polarization intact, Bob receives the state. **Direct transmission is a valid approach.** The practical challenge is that traveling photons can be lost or disturbed. Ordinary optical repeaters solve classical signal loss by detecting and regenerating bits. Doing this to an unknown qubit would normally destroy or fail to preserve its full state.

3.1 A small calculation showing the challenge

Optical fiber attenuation is often reported in decibels, abbreviated dB. For an idealized telecom-wavelength example, suppose the loss is 0.2 dB per kilometer. We do not need to know the engineering details of dB yet;

the transmitted intensity fraction is

$$T = 10^{-L_{\mathrm{dB}}/10}, \qquad L_{\mathrm{dB}} = (0.2\ \mathrm{dB/km})\, d. \tag{2}$$

At 50 km, the attenuation is 10 dB and $T = 10^{-1} = 0.10$. In this simplified model, only about 10% of single-photon attempts survive the fiber loss. At 100 km, the attenuation is 20 dB and $T = 0.01$, or about 1%. Real results also depend on wavelength, detectors, equipment, and other losses. The calculation is a way to understand the difficulty, not a prediction of a complete quantum network.

Approach	What can work	Main limitation
Send the physical photon	Bob receives the input if it survives undisturbed	Fiber loss and noise; a lost unknown input cannot simply be copied and retried
Measure, then send 0/1	Excellent for communicating the *observed* bit	Does not preserve an arbitrary input quantum state
Send a known preparation recipe	Works if Alice knows the state and Bob can prepare it	Not available for a single unknown state
Teleport using entanglement	Transfers the state after a shared Bell pair is ready	First establish and protect entanglement; perform accurate measurement and feed-forward

3.2 What improvement does teleportation offer?

A key change is when the difficult quantum link is used. Alice and Bob can attempt to distribute entanglement first. If a trial loses a photon, they may discard that trial and try again before introducing the valuable unknown input. Once a suitable Bell pair has been confirmed and stored, Alice performs the teleportation operations on her local qubits and sends two classical bits to Bob [8, 7].

Teleportation does not eliminate losses

The Bell pair still has to be created across a quantum connection, and those attempts can fail. Entanglement can decay during storage. The advantage is that a heralded successful resource can be prepared before consuming the unknown input, rather than risking that input on every trial. This does *not* automatically make teleportation faster, cheaper, or more reliable than direct transmission on every link.

Think and check. In the example above, why not simply resend Alice's unknown input after the photon is lost?
Explanation. Alice does not have unlimited independent copies of that single unknown input. She can repeat *entanglement preparation* before touching the input, but cannot repeat a lost unknown state by cloning it.

4 The intuition: what is shared, what is sent, and what changes

The phrase *the state is teleported* does not mean a physical object or signal instantly flies between laboratories. The operation has distinct stages. Understanding the order is as important as remembering the gates.

Stage	Alice's laboratory	Bob's laboratory
Before input	Obtains one half of a Bell pair, qubit A.	Obtains the other half, qubit B. Creating the pair may require a quantum channel.
Local operations	Combines input S with A using local gates, then measures both qubits.	Holds B while waiting.
Send two bits	Sends $m_S m_A$ over an ordinary classical channel.	Receives these two classical bits.
Recover output	No original unknown input remains at Alice.	Applies the selected X and/or Z; qubit B carries the input state.

1. **Before teleportation.** The two laboratories establish and keep a Bell pair. This earlier step may involve physical photons traveling through fiber or free space. Bob has qubit B; Alice has qubit A.
2. **When the input is ready.** Alice holds the unknown input qubit S. She needs only local operations on S and A, and then she measures both.
3. **Ordinary communication.** Alice sends two classical outcomes to Bob, by a conventional communication channel. These are instructions, not a numerical description of the state.
4. **After the message arrives.** Bob applies an input-independent correction to his existing qubit. It now carries the state of the original input, in an ideal protocol.

The cost of teleporting one arbitrary qubit

One initial input qubit, one *previously shared* entangled Bell pair (one qubit at each endpoint), two ordinary bits from Alice to Bob, and appropriate local gates and measurements. One Bell pair is consumed per ideal teleportation of one qubit.

Think and check. If Bob is 100 km away, can he complete the correction before Alice's classical message reaches him?
Explanation. No. Bob may hold or process his quantum register according to the protocol, but he cannot know which of the four corrections is required before receiving the corresponding classical information.

Checkpoint quiz 1 – Why do we need the protocol?

Q1.1. What can a classical router do to a bit that it cannot generally do to an unknown qubit?

Q1.2. Is direct transmission of a photon *impossible*, or does it face practical challenges? Give one challenge.

Q1.3. True or false. If a successful Bell pair has been prepared, Bob can recover the input using only Alice's two classical bits and his local correction, without the input photon traveling to Bob at that moment.

Answers for all checkpoint quizzes appear in the answer-key section.

Part II. How teleportation works

Review only the mathematical tools we need, then derive and verify the four possible outputs.

5 The input quantum state and its probabilities

A **qubit** is a quantum system with two reference states. A photon can encode a qubit in its polarization. For this lecture, the two reference states are

$$|0\rangle = \text{horizontal polarization}, \qquad |1\rangle = \text{vertical polarization}. \tag{3}$$

The notation $|0\rangle$ is read "ket zero". The vertical bar and angle bracket tell us that we are writing a quantum state, rather than an ordinary number.
A **pure state** is described by a single ket. A probabilistic mixture of possible states needs a more general description, introduced in Appendix B. An arbitrary pure qubit state can be written as

$$\boxed{|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \qquad |\alpha|^2 + |\beta|^2 = 1.} \tag{4}$$

The symbol ψ, pronounced "psi", is just the name of the state. The numbers α and β, pronounced "alpha" and "beta", are **probability amplitudes**. They may be real or complex. A complex number can contain i, where $i^2 = -1$. Its squared magnitude is always a nonnegative real number.
Measuring in the **computational basis**, also called the Z **basis**, asks which of the reference outcomes 0 or 1 occurs. The probabilities are

$$P(0) = |\alpha|^2, \qquad P(1) = |\beta|^2. \tag{5}$$

One measurement gives one outcome. It does not display α and β on a screen.

Example 1. Amplitudes are not probabilities

Consider

$$|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle. \tag{6}$$

The probabilities are $P(0) = 3/4$ and $P(1) = 1/4$. They add to 1. The coefficient $1/2$ on $|1\rangle$ gives a probability of $1/4$, not $1/2$.

5.1 Why probabilities alone are not the whole state

Define the **plus** and **minus** states by

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \qquad |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}. \tag{7}$$

Both give 0 and 1 with equal probability in a direct Z-basis measurement. Their difference is the sign between the two terms. This is a difference in **relative phase**, and it changes how amplitudes interfere when later gates are applied.

What successful teleportation must preserve

Bob must receive the entire state, including its relative phase. Reproducing only the probabilities of one measurement is not enough. We will explicitly check the difference between $|+\rangle$ and $|-\rangle$.

Think and check. If Alice measures $|+\rangle$ once and gets 0, has she learned that the input was $|0\rangle$?
Explanation. No. The outcome 0 is possible for $|+\rangle$, $|-\rangle$, and many other states. A possible measurement outcome does not uniquely identify the state that produced it.

6 Three gates that control amplitudes and phase

A **gate** is a controlled operation that changes a quantum state. For our calculations, a gate acts on each term of a superposition and the results are added. This property is called **linearity**.

Gate	Action on reference states	Action on the input state
X	$X\|0\rangle = \|1\rangle$, $X\|1\rangle = \|0\rangle$	$X\|\psi\rangle = \beta\|0\rangle + \alpha\|1\rangle$
Z	$Z\|0\rangle = \|0\rangle$, $Z\|1\rangle = -\|1\rangle$	$Z\|\psi\rangle = \alpha\|0\rangle - \beta\|1\rangle$
H	$H\|0\rangle = \|+\rangle$, $H\|1\rangle = \|-\rangle$	$H\|\psi\rangle = \alpha\|+\rangle + \beta\|-\rangle$

X exchanges the amplitudes of the two basis states. Z changes the sign of the $|1\rangle$ term relative to the $|0\rangle$ term. H, the **Hadamard gate**, converts between the 0/1 basis and the plus/minus basis.

Example 2. Why $H|+\rangle = |0\rangle$

Apply H to both terms, then combine like terms.

$$\begin{aligned} H|+\rangle &= \frac{H|0\rangle + H|1\rangle}{\sqrt{2}} \\ &= \frac{(|0\rangle + |1\rangle) + (|0\rangle - |1\rangle)}{2} \\ &= \frac{2|0\rangle + 0|1\rangle}{2} = |0\rangle. \end{aligned} \tag{8}$$

The two contributions to $|1\rangle$ cancel. The contributions to $|0\rangle$ add.

For the minus state, the minus sign must multiply the *whole* second bracket.

$$\begin{aligned} H|-\rangle &= \frac{H|0\rangle - H|1\rangle}{\sqrt{2}} \\ &= \frac{(|0\rangle + |1\rangle) - (|0\rangle - |1\rangle)}{2} \\ &= \frac{0|0\rangle + 2|1\rangle}{2} = |1\rangle. \end{aligned} \tag{9}$$

This explains how two states with identical direct 0/1 probabilities can be distinguished by a later operation. Apply H, then measure.

6.1 Undoing a change

Each of these gates undoes itself when applied twice.

$$X^2 = Z^2 = H^2 = I. \tag{10}$$

Here, I is the **identity operation**, which leaves the state unchanged. For example, two X gates exchange the amplitudes and then exchange them back.

Reading the order of gates

In $ZX|\psi\rangle$, the rightmost gate acts first. Read this as "apply X, then apply Z". In a circuit diagram, time runs from left to right. We will state Bob's correction in words as well as symbols.

Think and check. What does Z do to $|+\rangle$, and how can Bob undo it?
Explanation. It produces $|-\rangle$. A second Z gives $|+\rangle$ again.

7 Why measurement and CNOT cannot copy the state

7.1 Why not measure and send the result?

Suppose Alice receives one photon in an unknown state. A direct measurement returns 0 or 1. That single bit cannot tell Bob the original amplitudes and relative phase. In particular, both $|+\rangle$ and $|-\rangle$ could have produced exactly the same outcome.
Preparing many fresh, identically prepared inputs and measuring them in several bases can estimate a state. Alice does not have that supply of copies in the teleportation task. Repeatedly measuring the same already-measured photon does not provide fresh copies of its original unknown state.
If Alice already knows a preparation recipe, she can communicate it and Bob can prepare the known state. Teleportation addresses the more demanding case in which the arbitrary input need not be known to either person.

7.2 What CNOT does

The **controlled-NOT gate**, abbreviated CNOT, acts on two qubits. The first is the **control**; the second is the **target**. It flips the target when the control is 1 and leaves the target unchanged when the control is 0.

Control	Target	Input	Output
0	0	$\|00\rangle$	$\|00\rangle$
0	1	$\|01\rangle$	$\|01\rangle$
1	0	$\|10\rangle$	$\|11\rangle$
1	1	$\|11\rangle$	$\|10\rangle$

The notation $|01\rangle$ means "the first qubit is in $|0\rangle$, and the second is in $|1\rangle$". It does not mean one qubit has the value zero-one.

Example 3. CNOT does not clone $|+\rangle$

For a target initially in $|0\rangle$, CNOT copies the labels of the basis inputs $|0\rangle$ and $|1\rangle$. For a superposition, linearity gives

$$|+\rangle|0\rangle = \frac{|00\rangle + |10\rangle}{\sqrt{2}} \xrightarrow{\text{CNOT}} \frac{|00\rangle + |11\rangle}{\sqrt{2}}. \tag{11}$$

Two independent copies of $|+\rangle$ would instead have the state

$$|+\rangle|+\rangle = \frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2}. \tag{12}$$

These are different. The CNOT output gives only 00 and 11 when both qubits are measured in the Z basis. Two copies of $|+\rangle$ give all four outcomes equally often.

The **no-cloning theorem** states that a universal procedure cannot perfectly copy every arbitrary unknown quantum state while retaining the original [2]. The example above shows why CNOT is not such a copier; it is not, by itself, a proof covering every possible device.

Think and check. Does no-cloning prevent a laboratory from preparing many horizontally polarized photons?
Explanation. No. Those are repeated preparations of a known state. The restriction concerns a universal copier for arbitrary unknown inputs.

Checkpoint quiz 2 – Qubits, gates, and copying

Q2.1. What probabilities result from measuring $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ in the 0/1 basis?

Q2.2. Is $\text{CNOT}(|+\rangle|0\rangle)$ equal to $|+\rangle|+\rangle$? Explain by writing one term that appears in only one output.

Q2.3. What are the roles of H, X, and Z in one sentence each?

8 Preparing the Bell pair and assigning qubit roles

Alice and Bob begin with a shared **Bell pair**, a particular entangled two-qubit state.

$$|\Phi^+\rangle_{AB} = \frac{|00\rangle_{AB} + |11\rangle_{AB}}{\sqrt{2}}. \tag{13}$$

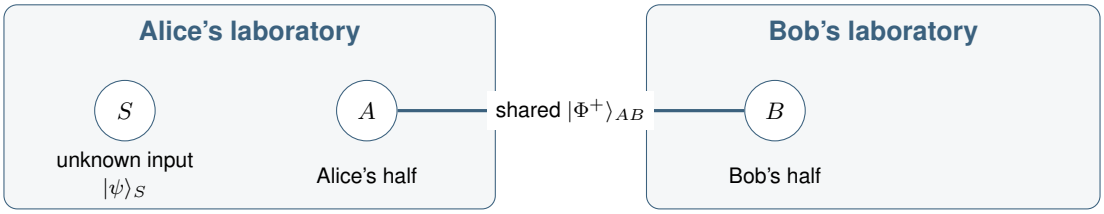
The superscript $+$ distinguishes this Bell state from three other Bell states. It is part of the name, not an instruction to add something.
Entangled means that the joint pure state cannot be written as a separate pure state for A multiplied by a separate pure state for B. The pair is described jointly. It is not a pair of independent photons, each secretly carrying $|+\rangle$.

8.1 Preparing the pair

Start with $|00\rangle_{AB}$, apply H to A, and then apply CNOT with A as control and B as target.

$$|00\rangle_{AB} \xrightarrow{H_A} \frac{|00\rangle_{AB} + |10\rangle_{AB}}{\sqrt{2}} \xrightarrow{\text{CNOT}_{A\to B}} \frac{|00\rangle_{AB} + |11\rangle_{AB}}{\sqrt{2}}. \tag{14}$$

This is a preparation circuit. In a simple picture, the pair is prepared together and then one member is physically distributed to each laboratory. Other physical methods can distribute entanglement, but that resource must already be available for this protocol.



The connecting line labels an entangled relationship. It is not a wire that carries an instantaneous message.

Qubit	Holder	Role
S	Alice	Source qubit carrying $\|\psi\rangle = \alpha\|0\rangle + \beta\|1\rangle$.
A	Alice	Alice's member of the shared Bell pair.
B	Bob	Bob's member of the pair; it will carry the transferred state.

The order we use everywhere

Every three-qubit string is written in the order $|SAB\rangle$. Thus $|101\rangle$ means $S = 1$, $A = 0$, and $B = 1$. Alice's classical message is always ordered $m_S m_A$, where m_S comes from measuring S and m_A comes from measuring A.

Think and check. Who holds two qubits at the start of the protocol, and who holds one?
Explanation. Alice holds S and A. Bob holds B. Alice's CNOT during teleportation will act only on her own two qubits.

9 The teleportation circuit: first read it, then calculate it

With the Bell pair already shared, Alice and Bob follow these steps [1, 5]. Figure 2 provides a conceptual roadmap before we work through the circuit algebraically.

1. **Alice couples her two qubits.** She applies $\mathrm{CNOT}_{S\to A}$. The input S is the control; A is the target.
2. **Alice applies H to S.** This makes the phase information contribute to her later measurement branches.
3. **Alice measures S and A in the Z basis.** She obtains two classical bits, m_S and m_A.
4. **Alice sends $m_S m_A$ to Bob.** An ordinary classical channel carries these bits.
5. **Bob corrects B.** If $m_A = 1$, he applies X. Then, if $m_S = 1$, he applies Z. A 0 means to skip that gate.

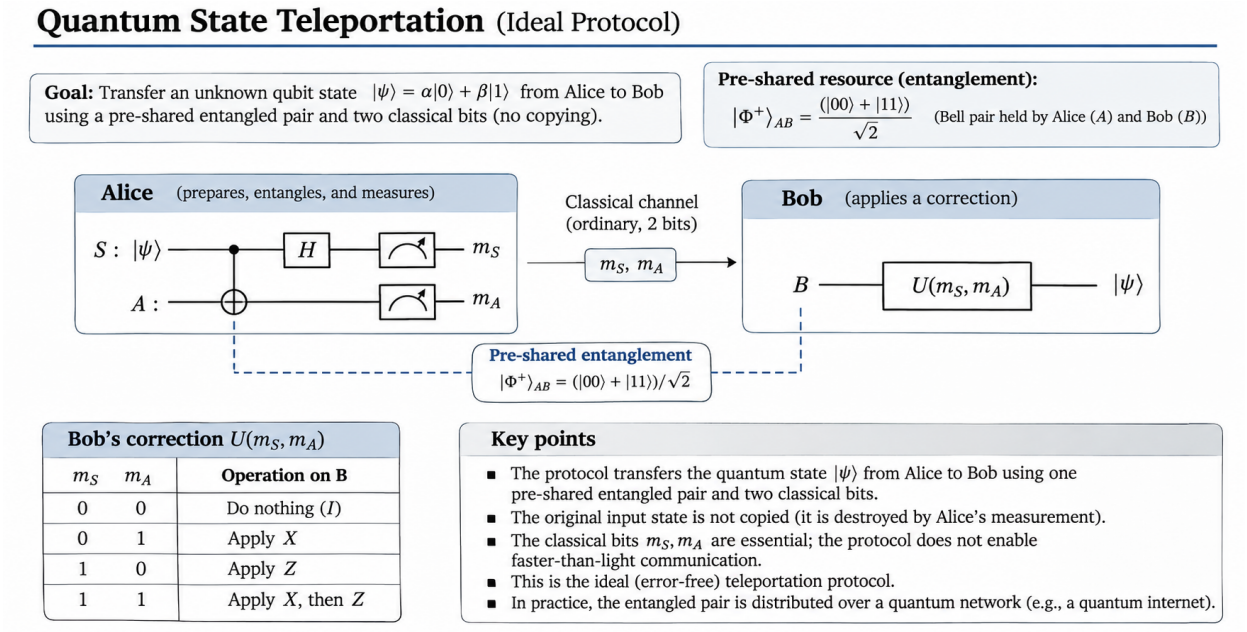
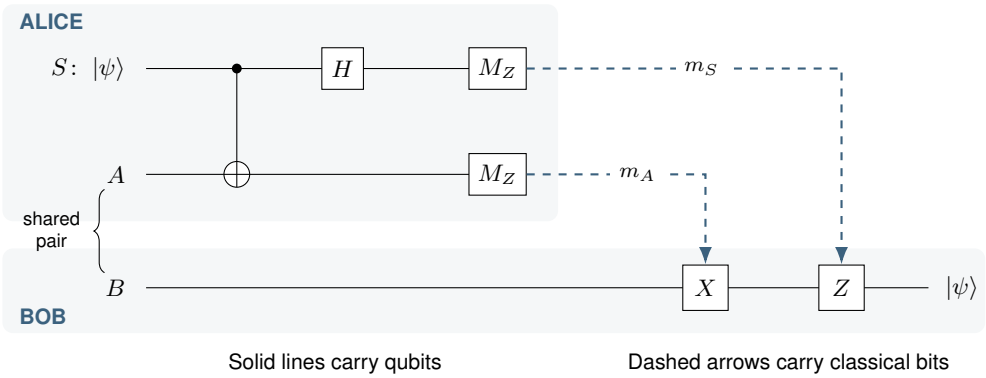


Figure 2: Ideal quantum-state teleportation with explicit classical measurement-bit convention. Alice holds input S and her half A of the pre-shared Bell pair; Bob holds B. After $\mathrm{CNOT}_{S\to A}$ and H_S, Alice measures S, A to obtain (m_S, m_A). Bob applies X if $m_A = 1$ and then Z if $m_S = 1$. The case 11 is the operation ZX acting on B (the order shown as "X, then Z"). No cloning or faster-than-light signaling occurs.



M_Z means a measurement in the 0/1 basis. Each dashed arrow controls whether the receiving gate is applied. There is no quantum gate between the separated laboratories in this part of the protocol.

m_S	m_A	Bob's action, in time order
0	0	Do nothing
0	1	Apply X
1	0	Apply Z
1	1	Apply X, then Z

Read one branch first

Suppose the input is $|+\rangle$ and Alice obtains 10. We will derive that Bob's state in this branch is $|-\rangle$. He applies Z, which changes $|-\rangle$ back into $|+\rangle$. Bob does not need to know the input to choose the gate. He needs only the measurement bits.

Keep these two rules attached to the qubit names

The measurement of A controls X. The measurement of S controls Z. A source that reverses the written order of the classical bits may display a different-looking table while describing the same protocol.

10 Full derivation: one gate at a time

10.1 Step 1. Write the starting state

The input and the Bell pair initially form the combined state

$$|\Psi_0\rangle_{SAB} = \underbrace{(\alpha|0\rangle + \beta|1\rangle)_S}_{\text{unknown input}} \otimes \underbrace{\frac{(|00\rangle + |11\rangle)_{AB}}{\sqrt{2}}}_{\text{shared pair}} . \tag{15}$$

The symbol $\otimes$ is the **tensor product**. Here it combines the state of one qubit with the state of the other two. Distribute the brackets as in ordinary algebra and join the labels in the fixed order S, A, B. For example, $|1\rangle_S \otimes |00\rangle_{AB} = |100\rangle_{SAB}$.

$$|\Psi_0\rangle = \frac{1}{\sqrt{2}} \left(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle\right) . \tag{16}$$

The capital symbol Ψ names the state of all three qubits. The lower-case ψ still names the input state we want Bob to receive.

10.2 Step 2. Apply $\mathrm{CNOT}_{S\to A}$

Look at the first digit. If it is 1, flip the second digit. Leave the third digit alone.

$$\begin{aligned}
|000\rangle &\longrightarrow |000\rangle \qquad S=0, \text{ so } A \text{ does not flip},\\
|011\rangle &\longrightarrow |011\rangle \qquad S=0, \text{ so } A \text{ does not flip},\\
|100\rangle &\longrightarrow |110\rangle \qquad S=1, \text{ so } A \text{ flips},\\
|111\rangle &\longrightarrow |101\rangle \qquad S=1, \text{ so } A \text{ flips}.
\end{aligned} \tag{17}$$

The state becomes

$$\boxed{|\Psi_1\rangle = \frac{1}{\sqrt{2}}\left(\alpha|000\rangle + \alpha|011\rangle + \beta|110\rangle + \beta|101\rangle\right).} \tag{18}$$

10.3 Step 3. Apply H to S

Replace the first digit using $H|0\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ or $H|1\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$. For example,

$$|110\rangle \xrightarrow{H_S} \frac{|010\rangle - |110\rangle}{\sqrt{2}}. \tag{19}$$

The original factor $1/\sqrt{2}$ and the new factor $1/\sqrt{2}$ multiply to $1/2$. Applying the gate to all four terms gives

$$\begin{aligned}
|\Psi_2\rangle = \frac{1}{2}\big(&\alpha|000\rangle + \alpha|100\rangle + \alpha|011\rangle + \alpha|111\rangle\\
&+ \beta|010\rangle - \beta|110\rangle + \beta|001\rangle - \beta|101\rangle\big).
\end{aligned} \tag{20}$$

Think and check. Why does the $\beta|110\rangle$ term produce one minus sign after H?
Explanation. Its first qubit is $|1\rangle$, and $H|1\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$. The other two qubits keep their labels.

11 The four measurement outcomes and Bob's corrections

Alice will measure the first two qubits. Group the eight terms in Eq. (20) according to the first two digits.

$$\boxed{\begin{aligned}
|\Psi_2\rangle = \frac{1}{2}\big[&|00\rangle_{SA}(\alpha|0\rangle + \beta|1\rangle)_B\\
&+|01\rangle_{SA}(\beta|0\rangle + \alpha|1\rangle)_B\\
&+|10\rangle_{SA}(\alpha|0\rangle - \beta|1\rangle)_B\\
&+|11\rangle_{SA}(-\beta|0\rangle + \alpha|1\rangle)_B\big].
\end{aligned}} \tag{21}$$

11.1 How to read one row

Suppose Alice gets 01, meaning $m_S = 0$ and $m_A = 1$. Select the row beginning with $|01\rangle_{SA}$. Before normalization, its Bob component is

$$\frac{1}{2}(\beta|0\rangle + \alpha|1\rangle). \tag{22}$$

The probability of this branch is its squared norm, the sum of its squared amplitude magnitudes.

$$P(01) = \left|\frac{\beta}{2}\right|^2 + \left|\frac{\alpha}{2}\right|^2 = \frac{|\alpha|^2 + |\beta|^2}{4} = \frac{1}{4}. \tag{23}$$

Once that outcome is known to have occurred, divide the branch by $\sqrt{1/4} = 1/2$. Bob's normalized conditional state is $\beta|0\rangle + \alpha|1\rangle$. The amplitudes are exchanged, so X exchanges them back.

$m_S m_A$	Bob's normalized state	Short form	Bob's correction
00	$\alpha\|0\rangle + \beta\|1\rangle$	$\|\psi\rangle$	I
01	$\beta\|0\rangle + \alpha\|1\rangle$	$X\|\psi\rangle$	X
10	$\alpha\|0\rangle - \beta\|1\rangle$	$Z\|\psi\rangle$	Z
11	$-\beta\|0\rangle + \alpha\|1\rangle$	$XZ\|\psi\rangle$	ZX, meaning X then Z

11.2 Why every outcome is correctable

Exchanging amplitudes or changing their signs does not change the norm. Each row of Eq. (21) has probability $1/4$, for every normalized input.

$$P(00) = P(01) = P(10) = P(11) = \frac{1}{4}. \tag{24}$$

The 11 branch is the one most likely to cause a sign error. Correct it explicitly.

$$-\beta|0\rangle + \alpha|1\rangle \xrightarrow{X} \alpha|0\rangle - \beta|1\rangle \xrightarrow{Z} \alpha|0\rangle + \beta|1\rangle. \tag{25}$$

Random outcome, reliable recovery

In the ideal protocol, the measurement result is random, but each result has a valid correction. With all four corrections available, teleportation succeeds for every branch. The probability $1/4$ of the 00 branch is not the success probability of the full protocol.

12 Worked example: unequal amplitudes

Let the input be the photon polarization state

$$|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle. \tag{26}$$

The state is normalized because $3/4 + 1/4 = 1$. A direct measurement of the original input would give 0 with probability $3/4$ and 1 with probability $1/4$.

12.1 Predict all four branches

Substitute $\alpha = \sqrt{3}/2$ and $\beta = 1/2$ into the general table.

Alice's result	Bob's state before correction	Correction
00	$\frac{\sqrt{3}}{2}\|0\rangle + \frac{1}{2}\|1\rangle$	Do nothing
01	$\frac{1}{2}\|0\rangle + \frac{\sqrt{3}}{2}\|1\rangle$	X
10	$\frac{\sqrt{3}}{2}\|0\rangle - \frac{1}{2}\|1\rangle$	Z
11	$-\frac{1}{2}\|0\rangle + \frac{\sqrt{3}}{2}\|1\rangle$	X, then Z

Example 4. Alice obtains 11

Bob starts in the conditional state

$$|\psi_{11}\rangle = -\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle. \tag{27}$$

First he applies X, because $m_A = 1$.

$$X|\psi_{11}\rangle = -\frac{1}{2}|1\rangle + \frac{\sqrt{3}}{2}|0\rangle = \frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle. \tag{28}$$

Then he applies Z, because $m_S = 1$.

$$ZX|\psi_{11}\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle = |\psi\rangle. \tag{29}$$

The output amplitudes are exactly the input amplitudes.

12.2 Keep three different probabilities separate

Question	Answer
What is the probability that Alice obtains 11?	$1/4$
Given Alice obtains 11, what is the probability that Bob measures 0 after correction?	$3/4$
What is the joint probability of Alice obtaining 11 and Bob then measuring 0 after correction?	$(1/4)(3/4) = 3/16$

Alice's four results are equally likely even though the input has unequal amplitudes. Her message tells Bob which change to undo; it does not report the input's 0/1 probabilities.

13 Worked examples: relative phase and complex amplitudes

Example 5. Teleporting $|+\rangle$ when Alice obtains 10

For an input $|\psi\rangle = |+\rangle$, the 10 branch gives

$$|\psi_{10}\rangle = Z|+\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} = |-\rangle. \tag{30}$$

Bob receives 10, applies Z, and obtains

$$Z|\psi_{10}\rangle = Z|-\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} = |+\rangle. \tag{31}$$

Without the correction, Bob has the wrong relative phase. With the correction, he has the input state.

13.1 A direct measurement can hide the error

Both $|+\rangle$ and $|-\rangle$ give equal probabilities of 0 and 1 when measured directly. To test the difference, apply H before measuring.

Bob's state	After H	Ideal measurement result
Correct state $|+\rangle$	$|0\rangle$	Always 0
Uncorrected state $|-\rangle$	$|1\rangle$	Always 1

Measuring after H is an X**-basis measurement**. The labels plus and minus are the two reference states in that basis. This is a practical way to observe phase information using an ordinary final 0/1 measurement.

Example 6. A complex amplitude is preserved too

Let

$$|\psi\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}}, \qquad i^2 = -1, \qquad |i|^2 = 1. \tag{32}$$

Suppose Alice obtains 01. Bob's state is

$$|\psi_{01}\rangle = X|\psi\rangle = \frac{i|0\rangle + |1\rangle}{\sqrt{2}}. \tag{33}$$

Bob applies X.

$$X|\psi_{01}\rangle = \frac{i|1\rangle + |0\rangle}{\sqrt{2}} = \frac{|0\rangle + i|1\rangle}{\sqrt{2}} = |\psi\rangle. \tag{34}$$

The relative factor i survives the protocol. The probabilities are real, $P(0) = P(1) = 1/2$; the imaginary coefficient is not an imaginary probability.

13.2 Relative phase and global phase

Changing $(|0\rangle + |1\rangle)/\sqrt{2}$ into $(|0\rangle - |1\rangle)/\sqrt{2}$ changes one term relative to the other and gives a different state. Multiplying *every* term by the same factor of magnitude 1, such as -1, is a **global phase** and does not change observable predictions.

For the 11 branch, applying Z and then X would give $-|\psi\rangle$, which is physically equivalent to $|\psi\rangle$. Our consistent convention is X and then Z, which recovers the written vector $|\psi\rangle$ exactly.

Checkpoint quiz 3 – Follow the teleportation circuit

Q3.1. Alice reports $m_S m_A = 10$. Which correction does Bob apply?

Q3.2. If $|\psi\rangle = \frac{3}{5}|0\rangle + \frac{4}{5}|1\rangle$ and Alice reports 01, write Bob's normalized state *before* correction.

Q3.3. For a normalized input, what is the probability of Alice's outcome 11 in the ideal protocol? Does the input affect this probability?

14 What is transferred, what is consumed, and what is preserved

14.1 What happens to Alice's input?

After Alice measures and Bob applies his correction, the three-qubit state conditioned on Alice's two results is

$$\boxed{|m_S\rangle_S \otimes |m_A\rangle_A \otimes |\psi\rangle_B.} \tag{35}$$

In the ideal circuit, Alice's measured qubits are in the measured basis states. They no longer hold the original arbitrary superposition. Bob's qubit carries that state. There is no second recoverable copy left at Alice's location.

This is a statement about the *state*. Some photon detectors absorb the photons they measure, but physical destruction of the input particle is not a required step in the abstract qubit protocol.

14.2 Did Bob receive Alice's photon?

No. Bob had B before Alice began the measurement procedure. The input carrier S was in Alice's laboratory. Teleportation transfers the quantum information to B, including its amplitudes and relative phase.

14.3 Why are two classical bits enough?

They do not encode numerical values for α and β. They identify one of four correction instructions. The full process also uses the pre-existing entangled pair.

For every input, Alice's four possible messages have the same probabilities, $1/4$ each. Seeing 01 does not tell Bob whether the input was $|0\rangle$, $|+\rangle$, or any other state. It tells him to apply X.

The classical message is an instruction

The message answers "Which correction should Bob apply?" It does not answer "What was the unknown quantum state?"

14.4 Can the shared pair be used again?

The original entanglement between A and B is consumed. Equation (35) is a product of Alice's measured states and Bob's output state. The pair AB is no longer a shared Bell pair. A new arbitrary input requires a fresh entangled pair in this standard protocol [5].

Resource for one input qubit	Standard ideal protocol
Input state to be transferred	1 qubit
Previously shared entanglement	1 Bell pair, containing 2 qubits
Classical message from Alice to Bob	2 bits
Output carrying the original state	Bob's existing qubit B
Remaining reusable Bell pairs from this run	0

Think and check. Can a third person create a copy merely by listening to Alice's two-bit message?
Explanation. No. The message alone does not specify the input. A listener does not hold Bob's particular share of the entangled resource.

15 Why the classical message is necessary

It is tempting to look at one row of Eq. (21) and say that Bob already has a state related to the input as soon as Alice measures. That conditional description is correct. The missing point is that **Bob does not yet know which row occurred**.
Alice cannot choose her random measurement result. Bob must receive her classical message before he can identify the required correction.

15.1 A calculation Bob can observe

Suppose Bob measures in the Z basis before receiving the message and without correcting. Average over Alice's four equally likely results.

$$\begin{aligned} P_B(0) &= \frac{1}{4}|\alpha|^2 + \frac{1}{4}|\beta|^2 + \frac{1}{4}|\alpha|^2 + \frac{1}{4}|\beta|^2 \\ &= \frac{|\alpha|^2 + |\beta|^2}{2} = \frac{1}{2}. \end{aligned} \tag{36}$$

Likewise, $P_B(1) = 1/2$. These probabilities do not depend on the input.

Example 7. Trying to signal with $|0\rangle$ or $|1\rangle$

Imagine Alice chooses $|0\rangle$ to mean "no" and $|1\rangle$ to mean "yes".

- For input $|0\rangle$, Bob's uncorrected state is physically $|0\rangle$ in two branches and $|1\rangle$ in two branches.
- For input $|1\rangle$, the roles of those branches interchange. Bob still sees $|0\rangle$ half the time and $|1\rangle$ half the time.

Without the message, Bob sees the same 50/50 statistics for either choice. He cannot read Alice's yes/no decision.

15.2 Could Bob choose a more clever measurement?

No. The calculation above illustrates the issue in one basis. A full calculation shows that Bob's unconditioned state is $I/2$, the equally mixed qubit state, for every input. This implies that *every* local measurement has input-independent statistics. Appendix B derives that result without assuming prior knowledge of density matrices.

Conditional state and accessible information are different

Someone who knows Alice's outcome can select a row of the table. Bob, without that outcome, must average over all four rows. Entanglement alone does not give him the missing information.

15.3 What must travel between the laboratories?

The entangled resource must first be physically established. During the protocol, Alice's two classical bits must reach Bob. Those bits cannot travel faster than light. Teleportation does not remove either requirement. Bob may store his qubit while the message is in transit. If he measures it first, he generally loses the quantum state that he was supposed to retain. Later receiving the message can help interpret correlations in the recorded data; it does not restore an already-measured arbitrary input.

Part III. What teleportation enables

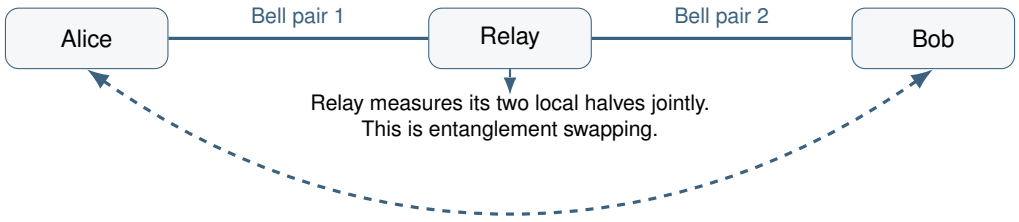
Return to quantum networks, repeaters, and modular quantum computers now that the protocol is clear.

16 Quantum repeaters and entanglement swapping

A **quantum network** connects devices that can prepare, store, manipulate, or measure quantum states. Unlike an ordinary internet, which primarily moves classical bit strings, a quantum network aims to distribute quantum entanglement and support tasks involving quantum states at different locations [7].

16.1 Why a quantum repeater is needed

Suppose Alice and Bob are far apart with an intermediate node, Relay. Rather than establish one fragile quantum link across the entire distance in one attempt, the network can try to establish entanglement on shorter segments.



Entanglement between Alice and Bob, after swapping and correction

16.2 Two different operations in one network

1. **Entanglement distribution.** Establish an entangled pair on the Alice–Relay link and another on the Relay–Bob link.
2. **Entanglement swapping.** The Relay performs a joint Bell-basis measurement on its two local qubits. Conditioned on that measurement and any needed corrections, Alice and Bob now share an entangled pair even though they did not directly interact. The Relay does not need to know the state of the future input.
3. **State teleportation.** Alice takes a *different*, arbitrary input qubit S and uses the newly shared Alice–Bob pair to teleport its state to Bob. She sends her two-bit result; Bob applies the correction.

Do not mix up these terms

Entanglement swapping extends an entangled resource to distant endpoints. **Quantum state teleportation** uses that resource to transfer a separate input state. The two protocols are related but not identical. Both may need classical outcome information.

A three-school research network

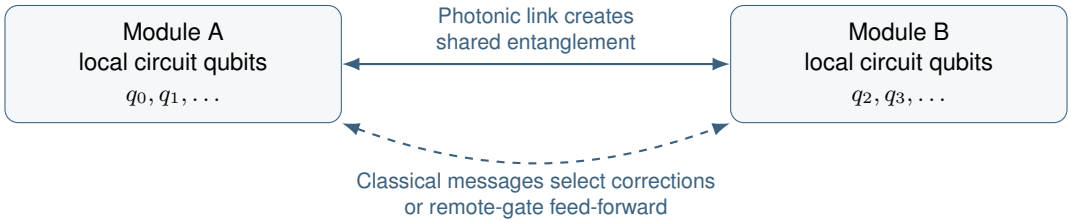
Imagine quantum devices at University A, a middle laboratory, and University B. Each neighbor pair can attempt to establish entanglement. The middle laboratory connects those two resources using entanglement swapping. University A can then teleport a state from one of its quantum memories

into a quantum memory at University B. The devices still require functioning quantum links, low-noise memories, synchronized control, and ordinary communication. A diagram of three connected institutions alone does not establish that all these capabilities exist.

Think and check. Can Alice teleport an arbitrary input if no suitable Alice–Bob Bell pair is ready?
Explanation. Not by the standard protocol. Alice must first obtain a suitable entangled resource while preserving the input.

17 Distributed quantum computing

In a conventional multicore computer, one processor can exchange ordinary data with another. A future **modular quantum computer** may have several smaller quantum processors connected by optical links instead of one enormous monolithic machine. To run a large quantum circuit, the modules must share quantum information or implement operations across modules [11].



17.1 Two related ways to use entanglement

1. **State teleportation.** Move a quantum state from a qubit in Module A into a qubit in Module B. After its transfer, Module B can continue processing it with local gates. This is the exact three-qubit protocol we derive in this lecture.
2. **Gate teleportation.** Use pre-shared entanglement and measurements to implement a logical gate between qubits kept in different modules. This is a related family of protocols; it does not mean that the entire input qubit must physically change modules. In 2025 researchers demonstrated a remote controlled-Z gate between two trapped-ion modules [11].

17.2 The subtle advantage of teleporting a correlated qubit

Suppose Alice's qubit S is entangled with another qubit R that remains at Alice. An example is

$$|\Phi^+\rangle_{RS} = \frac{|00\rangle_{RS} + |11\rangle_{RS}}{\sqrt{2}}. \tag{37}$$

When teleportation acts on S, ideal state transfer preserves its quantum correlations with R. After Bob's correction, the relevant state is

$$\boxed{|\Phi^+\rangle_{RB} = \frac{|00\rangle_{RB} + |11\rangle_{RB}}{\sqrt{2}}.} \tag{38}$$

This is important. Teleportation transfers more than the 0/1 measurement statistics of a standalone qubit. It also transfers the qubit's role in an entangled computation. The notation R denotes a reference qubit, not the Relay in the previous example; the two examples are separate.

Concept check

If the state of S is part of a larger computation, is it enough that Bob's measurements of B give 50/50 zeros and ones?
Answer. No. The correct *joint correlations* with other qubits must also be preserved. The Bell state above is not equivalent to two independent random bits.

18 Quantum memories, QKD, and security

18.1 Moving between different physical devices

Photons are useful for communication through fiber, while atoms, ions, and solid-state spins can serve as quantum memories or processors. Teleportation can transfer a state between different carriers when suitable entanglement and interfaces exist. Connecting distinct devices remains an engineering challenge [9].

18.2 Is teleportation the same as quantum key distribution?

No. In **quantum key distribution (QKD)**, such as the BB84 protocol from our earlier class, Alice and Bob aim to establish correlated secret classical bits and detect certain kinds of eavesdropping. In **teleportation**, the goal is to transfer an arbitrary quantum state using shared entanglement and two classical correction bits. A telecommunications network might use both ideas, but one protocol is not simply another name for the other.

Question	BB84 key distribution	Quantum teleportation
Main output	A shared classical secret key, after protocol checks and processing	A quantum state at the receiver
Uses quantum systems?	Yes	Yes
Needs classical messages?	Yes	Yes, including two correction bits per teleported qubit
Automatically encrypts your email?	No; a key must be used in a separate encryption scheme	No; the protocol is not an email-encryption system

18.3 Are Alice's two bits a secret code?

No. Alice's two measured results are each random, and for the ideal standard protocol the four two-bit messages occur with equal probability for any input. Those two bits do not contain a readable classical description of the unknown amplitudes. But **do not conclude that an arbitrary network implementation is automatically secure.** Real protocols must ensure their quantum resources have the expected properties, establish trusted identities, and address channel interference and errors. In particular, an adversary replacing or manipulating the entangled resource can change where the quantum state ends up. The standard protocol's information-theoretic statement assumes the proper shared Bell pair and intended endpoints.

Application scenario B. What is useful and what is not?

Useful. Alice has a qubit stored in a quantum memory, and Bob has a different quantum memory that must continue processing it. When they share a good Bell pair, teleportation moves the state to Bob's memory.
Not the application. Alice wishes to send Bob the password "CATS" as a classical message. Standard classical encryption and communication are appropriate; quantum teleportation provides no automatic speed or secrecy benefit for this ordinary task.

Think and check. A headline says "Teleportation makes messages instantly secret." Is that correct?
Explanation. No. It transfers quantum states with later classical communication, is not instantaneous, and is not an encryption scheme.

19 Real experiments and remaining engineering challenges

Researchers have built real teleportation experiments. The following results show the progress while also illustrating that laboratory demonstrations are not the same as a universal, large-scale quantum internet. All figures below refer to the cited experiments, not to the ideal circuit in our derivation.

Year	Experiment	What was actually reported
2022	Three-node network with solid-state qubits	Teleportation between *non-neighboring* quantum nodes, using entanglement swapping. Demonstrated a corrected-state fidelity above the classical benchmark in the reported setting [8].
2024	Urban fiber between a trapped ion and a remote photon	Quantum state teleportation from a trapped calcium-ion qubit to a telecom-wavelength photon over a 14.4 km deployed urban fiber link [9].
2024	Classical/quantum traffic sharing fiber	Teleportation over 30.2 km of *spooled laboratory fiber* carrying a 400-Gbps classical telecom signal; this showed coexistence, not a complete deployed quantum internet [10].
2025	Two trapped-ion processor modules	Gate teleportation of a controlled-Z operation between two trapped-ion modules separated by about 2 m, with about 86% average gate fidelity. This was *gate* teleportation, not a long-distance state-transfer experiment [11].

19.1 Why is it difficult outside an ideal circuit?

- **Entanglement generation.** The two endpoints must obtain a usable shared entangled resource despite photon loss and noise.
- **Quantum memories.** A successful Bell pair may need to be stored while another link or input becomes available. Stored states lose coherence over time.
- **Bell measurement.** Alice needs a joint measurement whose outcomes identify the required correction. A fully deterministic Bell measurement can be challenging in some optical implementations.
- **Timing and communication.** The receiver must obtain the relevant classical information and apply accurate feed-forward operations or account for them in later processing.
- **Testing fidelity.** Researchers need repeated, appropriately chosen input preparations and measurements to check whether the output matches the intended quantum states.

19.2 Successful delivery and correct state are different

A **success probability** describes how often a physical attempt yields an accepted usable output. **Fidelity** describes the quality of the accepted state compared with the target. For a known pure target state $|\psi\rangle$ and reconstructed output state ρ, one common fidelity definition is

$$F = \langle\psi|\rho|\psi\rangle, \qquad 0 \le F \le 1. \tag{39}$$

A device can have high fidelity on successful attempts but produce very few successful outputs. Or it can accept many outputs that have poor fidelity. These are different engineering problems. In the ideal circuit we study later, every branch can be corrected and has perfect fidelity. A real platform may not accept every Bell-measurement outcome or may experience additional noise.

Think about experimental claims

Suppose System A produces an accepted teleported state in 10% of attempts, with mean state fidelity 0.95. System B accepts 80% of attempts, with mean state fidelity 0.65. Can you call one *unambiguously* better?

Check. No. The comparison depends on the task, how fidelity and success are measured, and the required quality. Both numbers matter; fidelity alone or success probability alone is insufficient.

20 How to verify a quantum state was transferred

20.1 A useful classroom test

To test a device, the experimenter can prepare known inputs and check the outputs. The teleportation operations themselves still do not depend on knowing those input states. Repeat the entire experiment with fresh inputs and fresh Bell pairs.

Prepared input	Bob's check after correction	Ideal prediction
$\|0\rangle$	Measure directly in the Z basis	Always 0
$\|1\rangle$	Measure directly in the Z basis	Always 1
$\|+\rangle$	Apply H, then measure	Always 0
$\|-\rangle$	Apply H, then measure	Always 1

For $(|0\rangle + i|1\rangle)/\sqrt{2}$, first apply the phase-removal gate

$$D|0\rangle = |0\rangle, \qquad D|1\rangle = -i|1\rangle, \qquad D\frac{|0\rangle + i|1\rangle}{\sqrt{2}} = \frac{|0\rangle + |1\rangle}{\sqrt{2}}. \tag{40}$$

Then apply H and measure. The ideal result is 0. The gate D is commonly called $S^\dagger$; we use D here to avoid confusing the gate name with the source-qubit label S.
These checks detect errors in amplitudes and phase. Testing a few inputs does not prove every possible noisy device correct. Our ideal derivation covers every pure qubit input because it uses arbitrary α and β.

One successful run is not an experimental validation

A **shot** is one fresh preparation, execution, and measurement. Finite shot counts fluctuate, and real devices have errors. Inspect all four Alice outcomes and use measurements that reveal phase, as well as measurements that reveal 0/1 probabilities.

20.2 From a circuit to a laboratory

The ideal derivation assumes an exact Bell pair, accurate gates and measurements, and a correctly received message. Experiments must distribute entanglement, store qubits, implement the joint measurement, and apply or account for the correction. Loss and noise affect both output availability and quality.
Success probability asks how often a physical attempt supplies an accepted output. **Fidelity** describes how closely an output state matches the target. These are different quantities. For a known pure target, fidelity is the probability of obtaining the target outcome when measuring in a basis that contains that target. For target $|+\rangle$, apply H and count how often 0 occurs. An output $|+\rangle$ has fidelity 1; an output $|-\rangle$ has fidelity 0. This also shows why equal direct 0/1 probabilities do not establish a correct state. The ideal protocol has fidelity 1 for every corrected branch.

20.3 Historical context and why the idea matters

Charles H. Bennett, Gilles Brassard, Claude Crépeau, Richard Jozsa, Asher Peres, and William K. Wootters proposed the protocol in 1993 [1]. Dik Bouwmeester and colleagues demonstrated photonic teleportation in 1997 [3]. D. Boschi and colleagues published another early optical realization in 1998 [4]. These experiments established physical implementations; their apparatus and success conditions should not be confused with the ideal all-branches circuit.
Sections 16–19 describe uses in quantum networks and modular computing. State teleportation, entanglement swapping, and gate teleportation play different roles. Loss, memory errors, and classical communication delay remain engineering challenges.

Checkpoint quiz 4 – Interpret the network applications

Q4.1. In Alice–Relay–Bob, which operation connects neighboring Bell pairs? Which operation transfers a separate input state afterward?

Q4.2. If Bob has not yet received Alice's message, can he use his qubit to decode the input? Why not?

Q4.3. A paper reports 99% fidelity only for accepted trials. Name one additional measure needed to judge its usefulness as a network service.

21 Classroom activity and the main ideas

21.1 Activity. Follow the bits, not the unknown state

Work in groups of three. One student is Alice, one is Bob, and one checks the algebra. Use the input

$$|\psi\rangle = \frac{3}{5}|0\rangle + \frac{4}{5}|1\rangle. \tag{41}$$

The checker writes 00, 01, 10, and 11 on four slips and chooses one at random. Alice announces only the two-bit result. Bob writes the required correction without looking at the input coefficients. The checker uses the branch table to verify that the correction recovers the input.
Repeat with a different slip. This is a paper exercise about the correction rules. Randomly choosing a slip is not a physical simulation of entanglement.

Discuss before calculating

Why can Bob choose the right correction without knowing the coefficients $3/5$ and $4/5$? What part of the actual protocol is missing from the paper activity?

Discussion. The same correction rule works for every input. The paper activity assumes the conditional branch states; a real quantum device must produce those branches using the shared entangled pair and Alice's operations.

21.2 A compact vocabulary

Term	Meaning in this lecture
Quantum state	The description that predicts outcomes of operations and measurements.
Amplitude	A coefficient such as α or β; its squared magnitude gives a basis-outcome probability.
Relative phase	A phase difference between terms, such as the sign distinguishing $\|+\rangle$ from $\|-\rangle$.
Bell pair	Two qubits sharing one of the four entangled Bell states.
Classical bit	A recorded value 0 or 1 that can be sent by ordinary communication.
Conditional state	The state assigned after a particular measurement result is known.
Normalization	Scaling a state so its total probability is 1.
Correction	The X, Z, or combined operation selected by Alice's message.

Five statements you should be able to explain

1. Bob's output has the input state even when neither person knows that state.
2. Alice's result $m_S m_A$ tells Bob the correction, not the amplitudes.
3. m_A controls X, and m_S controls Z.
4. Alice loses the original arbitrary state and the Bell pair is consumed.

5. Bob needs classical communication to recover the state, so the protocol does not transmit usable information faster than light.

Part IV. Assess your understanding

Complete the self-check quiz first; use the extended recitation to deepen your reasoning. Worked answers follow all questions.

22 End-of-lecture quiz: Quantum teleportation and networking

Instructions. Try this short assessment *without* looking at the answer key. Part A checks core concepts. Part B checks your ability to follow the circuit and interpret a realistic network situation. The answer key follows the practice exercises.

Part A. Ten multiple-choice questions

1. The main goal of a quantum internet is to

A. replace every classical website with quantum software;
B. distribute entanglement and quantum information among networked devices;
C. copy unknown qubits in routers;
D. avoid any need for classical signals.

2. Which problem prevents a single unknown qubit from being reliably described by one measurement result?

A. A qubit is always measured as 2;
B. measurements reveal exact amplitudes;
C. measurement generally does not reveal the complete state and may disturb it;
D. classical links cannot carry bits.

3. The standard ideal teleportation of one qubit consumes

A. one Bell pair and two classical bits;
B. two Bell pairs and no classical bits;
C. one ordinary bit alone;
D. no quantum resource.

4. In this lecture, Alice holds S and A, and Bob holds B. The input state starts on

A. A;
B. B;
C. S;
D. all three separately.

5. For Alice's message $m_S m_A = 01$, Bob applies

A. nothing;
B. X;
C. Z;
D. H.

6. In ideal teleportation, Alice's four two-bit outcomes

A. reveal α and β;
B. occur only for $|0\rangle$;
C. are equally likely for any normalized pure input;
D. always equal 00.

7. Which is the correct description of *entanglement swapping*?

A. Measuring the unknown input in Z;
B. connecting entanglement across two links through a joint measurement at an intermediate node;
C. making two perfect copies of a qubit;
D. converting all qubits to classical keys.

8. Why can teleportation not send an intelligible message faster than light?

A. Bob has the correction information automatically;
B. the Bell pair has no quantum correlations;
C. Alice controls her random outcome;
D. Bob needs classical information to recover a usable state.

9. Both $|+\rangle$ and $|-\rangle$ give 50/50 in a direct Z-basis measurement. What distinguishes them ideally?

A. Apply H, then measure Z;
B. measure Z more slowly;
C. copy each once;
D. use no gate and only one shot.

10. A teleportation experiment reports excellent fidelity on accepted trials but very few accepted trials. A careful evaluation should report

A. fidelity alone;
B. only the fiber distance;
C. heralding/acceptance rate as well as fidelity and appropriate controls;
D. only the Alice outcome.

Part B. Short calculations and explanations

B1. For $|\psi\rangle = \sqrt{3}|0\rangle/2 + |1\rangle/2$, calculate the Z-basis probabilities and explain why the amplitudes are not the probabilities.

B2. Alice obtains 11 for the state in B1. Write Bob's conditional state before recovery and show the X and Z corrections in the proper order.

B3. Explain, in three or four sentences, how Alice, Relay, and Bob could use entanglement swapping followed by state teleportation.

23 Extended recitation exercises

Use the order $|SAB\rangle$ for the qubits and $m_S m_A$ for Alice's message. The shared pair is $|\Phi^+\rangle_{AB}$. Work out the reasoning before checking the solutions in Section 26.

1. Locate the qubits and read the message

Who holds S, A, and B? Which two qubits take part in Alice's CNOT, and which is the control? If Alice sends 10, state the result of each measurement and the gate Bob should apply.

2. Check an input state

Consider the input

$$|\psi\rangle = \frac{3}{5}|0\rangle + \frac{4}{5}|1\rangle. \tag{42}$$

Verify that it is normalized. Find the probabilities of 0 and 1 in a direct Z-basis measurement. Explain why $3/5$ and $4/5$ cannot themselves be the probabilities.

3. Follow the correct digits

Starting with the three-qubit basis state $|101\rangle_{SAB}$, apply $\mathrm{CNOT}_{S\to A}$, followed by H on S. Write the state after each gate. Identify the qubit whose label neither gate changes in this example.

4. Recover the 01 branch

Use the input in Eq. (42). Alice obtains 01. Write Bob's branch before normalization, calculate the probability of Alice's result, and then write Bob's normalized state. Show the correction that recovers the input.

5. Separate three probabilities

For the same input and the same 01 branch, calculate the probability of each event below.

(a) Alice obtains 01.

(b) Bob measures 0 after the correct recovery, given that Alice obtained 01.

(c) Alice obtains 01 and Bob subsequently measures 0 after recovery.

Which probabilities can change if the input amplitudes change?

6. Preserve a complex relative phase

The input is $|\psi\rangle = (|0\rangle + i|1\rangle)/\sqrt{2}$, and Alice obtains 11. Write Bob's normalized state. Apply X and then Z, showing both intermediate states. What is the probability of Alice obtaining 11, and does the i change that probability?

Recitation problems continued

The remaining questions test what the protocol means, including what a measurement can and cannot establish.

7. Detect a missed phase correction

Alice's input is $|+\rangle$, her result is 10, and Bob forgets to apply Z. Compare the direct Z-basis probabilities of Bob's actual state and the intended state. Now apply H before measuring each state. What outcome would each produce in the ideal case?

8. Diagnose a reversed message

Use the input in Eq. (42). Alice's actual result is 01, but Bob reads the two bits in reverse order and treats the message as 10. What state does he have before correction? What gate does he mistakenly apply, and what state results? Compare its probability of outcome 0 with the intended probability.

9. Remove the classical message

Again use Eq. (42). Alice performs her measurements, but Bob receives no result and applies no correction. List the probability that Bob measures 0 in each of the four possible branches. Average them using the branch probabilities. Would observing 0 allow Bob to know whether Alice obtained 00?

10. Test a proposed quantum copier

A student proposes putting $|-\rangle$ into the control of a CNOT and $|0\rangle$ into its target. Expand the input and calculate the output. Separately expand $|-\rangle|-\rangle$, the state of two copies. How could a joint Z-basis

measurement distinguish the predictions of the two states over repeated preparations?

11. Repair three explanations

Each statement below contains a mistake. Replace it with one accurate sentence and explain the mistake briefly.

(a) "The two classical bits tell Bob the input amplitudes."

(b) "Bob receives Alice's original photon."

(c) "The protocol succeeds only when Alice measures 00."

12. Count the resources

Alice wants to teleport 5 independent unknown qubit states, using the standard protocol once per input. How many shared Bell pairs are consumed, and how many classical bits must Alice send? Can the same already-used Bell pair teleport the next input? Explain why the resource count does not by itself require 15 qubits to be held at the same moment if the runs occur sequentially.

24 Applied recitation: choosing the right network primitive

These exercises test the reasons for using the protocol, not just the gate calculations. Explain each answer in plain words.

A1. Classical versus quantum. Alice wants to send Bob an ordinary text document. Alice later wants to transmit a single unknown qubit. Which task requires a quantum resource, and why?

A2. A lossy link. For the simplified fiber model in Equation (2), calculate the transmission fraction at 25 km. What *additional* physical resource must a teleportation link prepare, even if that input photon does not directly travel to Bob?

A3. Two-stage repeater. Alice and Bob do not share entanglement, but Alice–Relay and Relay–Bob do. Describe the operation that can create an Alice–Bob entangled pair, then name the separate operation that moves an arbitrary input state to Bob.

A4. An honest benchmark. An experimental report says "teleportation fidelity 92% on accepted trials" but gives no acceptance probability. What essential information is missing for comparing its communication usefulness?

A5. Computing application. Module A and Module B cannot perform direct two-qubit gates with each other. Name one situation that would use state teleportation and one that might use a form of gate teleportation.

A6. QKD is different. Is the output of BB84 a quantum state on Bob's processor or an agreed classical key? Is teleportation itself an encryption algorithm?

25 Answer key: checkpoint quizzes and final quiz

Checkpoint quiz 1

Q1.1 A classical router can read/copy/regenerate classical bits. A universal device cannot create an independent perfect copy of an arbitrary unknown quantum state. **Q1.2** Direct photon transmission is possible, but photons may be lost or disturbed. **Q1.3** True. Once a suitable shared pair is available, the unknown input need not travel across the remote link during the local Bell measurement and classical correction stage. Creating that pair has already required a physical quantum resource.

Checkpoint quiz 2

Q2.1 Both probabilities are $1/2$. **Q2.2** No. CNOT produces $(|00\rangle + |11\rangle)/\sqrt{2}$, while two copies produce $(|00\rangle + |01\rangle + |10\rangle + |11\rangle)/2$. The term $|01\rangle$ appears only in the latter. **Q2.3** H mixes the 0/1 basis into plus/minus superpositions; X exchanges $|0\rangle$ and $|1\rangle$; Z flips the sign of the $|1\rangle$ term.

Checkpoint quiz 3

Q3.1 Apply Z. **Q3.2** The state before correction is $(4/5)|0\rangle + (3/5)|1\rangle$; X restores the input. **Q3.3** $1/4$, independent of the normalized input state in the ideal protocol.

Checkpoint quiz 4

Q4.1 The relay performs entanglement swapping on its two local halves, establishing a remote entangled pair with appropriate classical correction bookkeeping; afterward Alice can use this pair for state teleportation. **Q4.2** No. Before receiving the classical message, Bob has an input-independent mixed qubit; he cannot select the needed correction or extract Alice's state. **Q4.3** Report the acceptance or success probability and ideally the delivered-state rate, alongside how the fidelity was measured.

End-of-lecture quiz, Part A

1	2	3	4	5	6	7	8	9	10
B	C	A	C	B	C	B	D	A	C

Why these answers? (1) A quantum network adds entanglement and state-transfer capabilities. (2) A single measurement is not a complete arbitrary-state description. (3) One Bell pair and two bits are the protocol resources. (4) S is the input. (5) The second bit controls X. (6) Alice's outcomes are uniform in the ideal circuit. (7) Swapping builds longer-distance entanglement. (8) Bob requires the ordinary message. (9) H maps plus/minus to different Z-basis outcomes. (10) Fidelity without acceptance or rate gives an incomplete picture.

End-of-lecture quiz, Part B – worked answers

B1. The amplitudes are $\alpha = \sqrt{3}/2$ and $\beta = 1/2$. The probabilities are $P(0) = |\alpha|^2 = 3/4$ and $P(1) = |\beta|^2 = 1/4$. Probability requires squaring the magnitude of each coefficient. **B2.** Alice's 11 branch is $-\beta|0\rangle + \alpha|1\rangle = -\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$. Applying X first gives $\frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle$. Applying Z next gives $\frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle = |\psi\rangle$. For 11 the operations must be ordered as X then Z in this convention. **B3.** Alice and Relay first share an entangled pair, and Relay and Bob share another. Relay measures its two local halves jointly, creating Alice–Bob entanglement through *entanglement swapping*, once measurement outcomes are appropriately accounted for. Alice uses the newly shared pair to teleport a separate unknown input state by measuring locally and sending Bob two bits. Bob applies the indicated correction. The repeater does not simply copy the unknown input.

26 Worked solutions to the extended recitation

1. Ownership, gate direction, and bit order

Alice holds S and A, while Bob holds B. Alice applies CNOT with S as control and A as target. Her message 10 means $m_S = 1$ and $m_A = 0$. Bob applies Z, because the measurement of S controls the phase correction. He skips X, because the measurement of A was 0. Both of Alice's gates act inside her own laboratory.

2. Normalize the state and square the amplitudes

The squared magnitudes add to 1.

$$\left|\frac{3}{5}\right|^2 + \left|\frac{4}{5}\right|^2 = \frac{9}{25} + \frac{16}{25} = 1. \tag{43}$$

Thus $P(0) = 9/25$ and $P(1) = 16/25$. The coefficients $3/5$ and $4/5$ are amplitudes. If we incorrectly treated them as probabilities, their sum would be $7/5$, which is greater than 1.

3. Apply each gate to its own qubits

The control S is 1, so CNOT flips A from 0 to 1. It does not change B.

$$|101\rangle_{SAB} \xrightarrow{\text{CNOT}_{S\to A}} |111\rangle_{SAB}. \tag{44}$$

Hadamard then acts on the first qubit only.

$$|111\rangle_{SAB} \xrightarrow{H_S} \frac{|011\rangle_{SAB} - |111\rangle_{SAB}}{\sqrt{2}}. \tag{45}$$

The third qubit, B, remains $|1\rangle$ in both terms. The minus sign comes from $H|1\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$.

4. Select, normalize, and correct the 01 branch

For outcome 01, Bob's component before normalization is

$$\frac{1}{2}\left(\frac{4}{5}|0\rangle + \frac{3}{5}|1\rangle\right) = \frac{2}{5}|0\rangle + \frac{3}{10}|1\rangle. \tag{46}$$

Its squared norm gives the probability of this outcome.

$$P(01) = \left|\frac{2}{5}\right|^2 + \left|\frac{3}{10}\right|^2 = \frac{16}{100} + \frac{9}{100} = \frac{1}{4}. \tag{47}$$

Divide the branch by $\sqrt{1/4} = 1/2$. Bob's normalized state is $(4/5)|0\rangle + (3/5)|1\rangle$. Because $m_A = 1$, he applies X.

$$X\left(\frac{4}{5}|0\rangle + \frac{3}{5}|1\rangle\right) = \frac{4}{5}|1\rangle + \frac{3}{5}|0\rangle = |\psi\rangle. \tag{48}$$

The correction exchanges the amplitudes back. The branch factor $1/2$ is used to calculate the branch probability; it is not retained in the normalized state after conditioning on that outcome.

5. Distinguish an outcome, a conditional probability, and a joint event

Every Alice outcome has probability $1/4$. After recovery, Bob has the original input, whose probability of 0 is $9/25$.

$$\begin{aligned} P(\text{Alice gets } 01) &= \frac{1}{4}, \\ P(\text{Bob gets } 0 \mid \text{Alice gets } 01, \text{ corrected}) &= \frac{9}{25}, \\ P(\text{Alice gets } 01 \text{ and Bob gets } 0 \text{ after recovery}) &= \frac{1}{4}\cdot\frac{9}{25} = \frac{9}{100}. \end{aligned} \tag{49}$$

The vertical bar means "given that". Changing the input can change the last two probabilities. Alice's branch probability remains $1/4$ for every normalized input in the ideal protocol.

6. Keep the complex phase through both corrections

Substitute $\alpha = 1/\sqrt{2}$ and $\beta = i/\sqrt{2}$ into the 11 branch, $-\beta|0\rangle + \alpha|1\rangle$.

$$\frac{-i|0\rangle + |1\rangle}{\sqrt{2}} \xrightarrow{X} \frac{|0\rangle - i|1\rangle}{\sqrt{2}} \xrightarrow{Z} \frac{|0\rangle + i|1\rangle}{\sqrt{2}} = |\psi\rangle. \tag{50}$$

The relative phase is restored exactly. Since $|i|^2 = 1$, the input and each normalized branch have norm 1. Including the branch factor $1/2$ gives

$$P(11) = \left|\frac{-i}{2\sqrt{2}}\right|^2 + \left|\frac{1}{2\sqrt{2}}\right|^2 = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}. \tag{51}$$

The complex phase changes the state, but it does not change Alice's four equally likely outcome probabilities.

7. Use a measurement that can reveal the missing phase

The actual state is $Z|+\rangle = |-\rangle$; the intended state is $|+\rangle$. Direct Z-basis measurements give $P(0) = P(1) = 1/2$ for both. Apply H first.

$$\begin{aligned} H|+\rangle &= \frac{(|0\rangle + |1\rangle) + (|0\rangle - |1\rangle)}{2} = |0\rangle, \\ H|-\rangle &= \frac{(|0\rangle + |1\rangle) - (|0\rangle - |1\rangle)}{2} = |1\rangle. \end{aligned} \tag{52}$$

The intended state now gives 0 with certainty; the incorrect state gives 1 with certainty. This ideal test exposes a phase error that direct 0/1 probabilities miss.

8. Trace the effect of reading the bits backward

The actual 01 outcome leaves Bob in $(4/5)|0\rangle + (3/5)|1\rangle$. Reading it as 10 makes him apply Z instead of X.

$$\frac{4}{5}|0\rangle + \frac{3}{5}|1\rangle \xrightarrow{Z} \frac{4}{5}|0\rangle - \frac{3}{5}|1\rangle. \tag{53}$$

This state has $P(0) = 16/25$, while the intended output has $P(0) = 9/25$. The incorrect result comes from swapping the roles of m_S and m_A, not from a failure of the teleportation identity.

9. Average the unknown branches

Before correction, the coefficient on $|0\rangle$ is $3/5$, $4/5$, $3/5$, or $-4/5$, depending on Alice's result.

Alice's result	00	01	10	11
Bob's probability of 0	$9/25$	$16/25$	$9/25$	$16/25$

Without Alice's message, Bob averages these possibilities using their equal probabilities.

$$P_B(0) = \frac{1}{4}\left(\frac{9}{25} + \frac{16}{25} + \frac{9}{25} + \frac{16}{25}\right) = \frac{1}{2}. \tag{54}$$

Observing 0 cannot identify the 00 branch. In this example, every branch can produce 0. More generally, Bob's state without Alice's result is the same completely mixed qubit for every input, so no measurement lets him read the input from his qubit alone. Alice cannot choose her random result, and Bob cannot select the 00 cases without learning her message.

10. Compare entanglement with two independent copies

The proposed CNOT acts as

$$|-\rangle|0\rangle = \frac{|00\rangle - |10\rangle}{\sqrt{2}} \xrightarrow{\text{CNOT}} \frac{|00\rangle - |11\rangle}{\sqrt{2}}. \tag{55}$$

Two copies would have the different state

$$|-\rangle|-\rangle = \frac{(|0\rangle - |1\rangle)(|0\rangle - |1\rangle)}{2} = \frac{|00\rangle - |01\rangle - |10\rangle + |11\rangle}{2}. \tag{56}$$

The CNOT output produces only 00 and 11, each with probability $1/2$. Two copies produce 00, 01, 10, and 11, each with probability $1/4$. Repeated joint measurements can reveal these different distributions. A single occurrence of 00 or 11 would not distinguish them. CNOT has created an entangled pair, not two copies of $|-\rangle$.

11. Replace each mistaken explanation

(a) The two classical bits identify Bob's correction. Every message has probability $1/4$, independent of the input amplitudes, so the message is not a description of the input state.
(b) Bob's existing photon acquires the input quantum state after the appropriate correction. Teleportation transfers the state to a different physical carrier; Alice's input photon does not travel to Bob in this protocol. Sharing the entangled pair already required a physical preparation and distribution process.
(c) All four branches succeed in the ideal protocol when Bob applies the corresponding correction. The 00 branch needs no gate, while 01, 10, and 11 need X, Z, and X followed by Z, respectively. A branch probability of $1/4$ is not the full protocol's success probability.

12. Count consumed resources separately from reusable equipment

Each independent input uses one shared Bell pair and two transmitted classical bits. Five inputs use **5 Bell pairs and 10 classical bits**. The measurements consume the original shared entanglement, so the same already-used pair cannot teleport the next input without new entanglement being established.
There are three qubit roles in each run, with two at Alice and one at Bob. Sequential runs need not hold all 15 roles simultaneously if an output is processed or transferred onward and suitable hardware is reset or replenished. Fresh entanglement is still needed for each run. If Bob must keep all five outputs available at once, he must provide storage for those five output qubits.

Worked answers to the applied recitation

A1. The text document is already classical bits and can be sent over an ordinary digital connection. An arbitrary unknown qubit cannot be fully learned from a single measurement and reconstructed from the outcome. The standard teleportation procedure needs entanglement plus ordinary correction bits.
A2. The loss is $L_{\mathrm{dB}} = 0.2(25) = 5$ dB. The transmission fraction is $T = 10^{-5/10} = 10^{-0.5} \approx 0.316$, or 31.6%. Teleportation must first prepare and preserve a shared entangled Bell pair. That step may also involve lost photons; a fiber connection does not become lossless just because teleportation is used.
A3. Relay can make a joint Bell-basis measurement on its halves of the two short-link pairs, using *entanglement swapping* to create an Alice–Bob pair, with suitable classical tracking or corrections. Alice can then use that pair in *state teleportation* to transfer the separate input qubit to Bob.
A4. We need the probability that an attempt is accepted, the attempt or repetition rate, and how fidelity was defined and estimated. High conditional fidelity by itself is not a complete measure of a useful communication link.
A5. In state teleportation, Module A transfers the state of one of its circuit qubits to Module B, so that B can continue with local processing. In gate teleportation, the devices use entanglement and classical feed-forward to realize a logical gate coupling qubits that stay at distinct modules. The exact operations and resource costs depend on the chosen gate-teleportation protocol.
A6. BB84 aims to establish secret *classical* key bits after the necessary checks and processing. State teleportation makes a quantum output state at Bob. It is not, by itself, an algorithm that encrypts arbitrary classical data.

A Bell measurement: a closer look

The main lecture used ordinary 0/1 measurements after CNOT and H. Taken together, those operations implement a **Bell-basis measurement** on Alice's two qubits.
A **basis** is a set of reference states used to express other states. An **orthonormal basis** consists of

normalized, mutually perpendicular state vectors. The four two-qubit Bell states are

$$|\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \quad |\Phi^-\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}},$$
$$|\Psi^+\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}, \quad |\Psi^-\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}. \tag{57}$$

These capital Φ and Ψ symbols name Bell states. They are not extra qubits. In the following table the two qubits are Alice's S, A, in that order.

Bell state of SA	After $\mathrm{CNOT}_{S\to A}$	After H_S	Measured $m_S m_A$
$\|\Phi^+\rangle$	$\|+\rangle\|0\rangle$	$\|00\rangle$	00
$\|\Phi^-\rangle$	$\|-\rangle\|0\rangle$	$\|10\rangle$	10
$\|\Psi^+\rangle$	$\|+\rangle\|1\rangle$	$\|01\rangle$	01
$\|\Psi^-\rangle$	$\|-\rangle\|1\rangle$	$\|11\rangle$	11

Each Bell state is mapped to a different ordinary bit string. Measuring those strings after the gates is equivalent to distinguishing the Bell states before the gates.

Follow the $|\Phi^-\rangle$ row

$$\frac{|00\rangle - |11\rangle}{\sqrt{2}} \xrightarrow{\text{CNOT}} \frac{|00\rangle - |10\rangle}{\sqrt{2}} = |-\rangle|0\rangle \xrightarrow{H_S} |10\rangle. \tag{58}$$

The outcome is 10 with certainty if the measured two-qubit input to this circuit is $|\Phi^-\rangle$. In teleportation, Alice's pair is part of the three-qubit state, so all four outcomes instead occur with equal probability.

A compact formula for the correction

The four branches can be written as one expression.

$$|\psi_{m_S m_A}\rangle = X^{m_A} Z^{m_S} |\psi\rangle, \qquad m_S, m_A \in \{0, 1\}. \tag{59}$$

A zero exponent means the identity; a one exponent means to apply the gate. Bob's correction is $Z^{m_S} X^{m_A}$, so

$$\begin{aligned} Z^{m_S} X^{m_A} |\psi_{m_S m_A}\rangle &= Z^{m_S} X^{m_A} X^{m_A} Z^{m_S} |\psi\rangle \\ &= Z^{m_S} I Z^{m_S} |\psi\rangle = |\psi\rangle. \end{aligned} \tag{60}$$

This proves recovery without having to choose numerical values for α or β. The same linear action also preserves correlations when the input is entangled with another system. In that case, the local input cannot in general be described by a pure state $|\psi\rangle$ alone; the protocol acts on its part of the larger joint state [5].

B Optional mathematical appendix: no-signaling

This appendix explains why *any* measurement at Bob's location, before he uses Alice's message, has input-independent statistics.

Step 1. Write a state as a column and its bra as a row

For a pure state, define

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \qquad \langle\psi| = \begin{pmatrix} \alpha^* & \beta^* \end{pmatrix}. \tag{61}$$

The star means **complex conjugation**, which changes i to $-i$. A bra is the conjugated row associated with a ket. Multiplying the column by the row gives the **density matrix**

$$\rho = |\psi\rangle\langle\psi| = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \beta\alpha^* & |\beta|^2 \end{pmatrix}. \tag{62}$$

The diagonal entries are the 0/1 probabilities. The other entries retain phase information. A mixture of possible states is described by averaging their density matrices with their probabilities.

Step 2. Average the four possible branches

Let $a = |\alpha|^2$, $b = |\beta|^2$, and $c = \alpha\beta^*$, so $a + b = 1$. The gate matrices are

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \rho = \begin{pmatrix} a & c \\ c^* & b \end{pmatrix}. \tag{63}$$

Since Bob does not know Alice's result, his density matrix is

$$\rho_B = \frac{1}{4}\big(\rho + X\rho X + Z\rho Z + XZ\rho ZX\big). \tag{64}$$

For an operation U, a density matrix changes as $U\rho U^\dagger$, where the dagger means conjugate transpose. In the last term, $(XZ)^\dagger = ZX$. The order on the right matters.
Pair the terms so the cancellation is visible.

$$\begin{aligned} \rho + Z\rho Z &= \begin{pmatrix} 2a & 0 \\ 0 & 2b \end{pmatrix}, \\ X\rho X + XZ\rho ZX &= \begin{pmatrix} 2b & 0 \\ 0 & 2a \end{pmatrix}. \end{aligned} \tag{65}$$

Adding and dividing by 4 yields

$$\boxed{\rho_B = \frac{1}{4}\begin{pmatrix} 2(a+b) & 0 \\ 0 & 2(a+b) \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} = \frac{I}{2}.} \tag{66}$$

No input coefficient remains. For a measurement outcome represented by any normalized basis vector $|u\rangle$,

$$P(u) = \langle u|\rho_B|u\rangle = \frac{1}{2}\langle u \mid u\rangle = \frac{1}{2}. \tag{67}$$

More general measurements also have probabilities determined entirely by the same input-independent matrix. Bob learns nothing about the input from his qubit alone until the classical outcome information becomes available.

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